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Strongly anisotropic diffusion problems; asymptotic analysis

Mihai Bostan

Aix Marseille Université, CNRS, Centrale Marseille, LATP, UMR 7353, 13453 Marseille, France

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Abstract

The subject matter of this paper concerns anisotropic diffusion equations: we consider heat equations whose diffusion matrices have disparate eigenvalues. We determine first and second order approximations, we study the well-posedness of them and establish convergence results. The analysis relies on averaging techniques, which have been used previously for studying transport equations whose advection fields have disparate components.

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1. Introduction

Many real life applications lead to highly anisotropic diffusion equations: flows in porous media, quasi-neutral plasmas, microscopic transport in magnetized plasmas [7], plasma thrusters, image processing [18,23], thermal properties of crystals [13,19]. In this paper we investigate the behavior of the solutions for heat equations whose diffusion becomes very high along some direction. We consider the problem

$$\partial_t u^\varepsilon - \operatorname{div}_y\big(D(y)\nabla_y u^\varepsilon\big) - \frac{1}{\varepsilon}\operatorname{div}_y\big(b(y)\otimes b(y)\nabla_y u^\varepsilon\big) = 0, \quad (t,y)\in\mathbb{R}_+\times\mathbb{R}^m, \tag{1}$$

E-mail address: bostan@cmi.univ-mrs.fr.

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$$u^\varepsilon(0,y) = u^\varepsilon_{\text{in}}(y), \quad y \in \mathbb{R}^m, \tag{2}$$

where $D(y) \in \mathcal{M}_m(\mathbb{R})$ and $b(y) \in \mathbb{R}^m$ are smooth given matrix field and vector field on $\mathbb{R}^m$, respectively. For any two vectors ξ, η, the notation $\xi \otimes \eta$ stands for the matrix whose entry (i,j) is $\xi_i \eta_j$, and for any two matrices A, B the notation $A : B$ stands for $\operatorname{trace}({}^t AB) = A_{ij}B_{ij}$ (using Einstein summation convention). We assume that at any $y \in \mathbb{R}^m$ the matrix $D(y)$ is symmetric and $D(y) + b(y) \otimes b(y)$ is positive definite

$$\begin{gathered} {}^tD(y) = D(y), \\ \exists d > 0 \quad \text{such that} \quad D(y)\xi \cdot \xi + \big(b(y)\cdot\xi\big)^2 \geqslant d|\xi|^2, \quad \xi \in \mathbb{R}^m,\ y \in \mathbb{R}^m. \end{gathered} \tag{3}$$

The vector field $b(y)$, to which the anisotropy is aligned, is supposed divergence free *i.e.*, $\operatorname{div}_y b = 0$. We intend to analyze the behavior of (1), (2) for small ε, let us say $0 < \varepsilon \leqslant 1$, in which cases $D(y) + \frac{1}{\varepsilon} b(y) \otimes b(y)$ remains positive definite

$$D(y)\xi \cdot \xi + \frac{1}{\varepsilon}\big(b(y)\cdot\xi\big)^2 \geqslant D(y)\xi\cdot\xi + \big(b(y)\cdot\xi\big)^2 \geqslant d|\xi|^2, \quad \xi\in\mathbb{R}^m,\ y\in\mathbb{R}^m. \tag{4}$$

If $(u^\varepsilon_{\text{in}})_\varepsilon$ remain in a bounded set of $L^2(\mathbb{R}^m)$, then $(u^\varepsilon)_\varepsilon$ remain in a bounded set of $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ since, for any $t \in \mathbb{R}_+$ we have, thanks to (4)

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^m} \big(u^\varepsilon(t,y)\big)^2\,\mathrm{d}y + d\int_0^t\int_{\mathbb{R}^m} \big|\nabla_y u^\varepsilon(s,y)\big|^2\,\mathrm{d}y\,\mathrm{d}s \\ &\quad\leqslant \frac{1}{2}\int_{\mathbb{R}^m}\big(u^\varepsilon(t,y)\big)^2\,\mathrm{d}y + \int_0^t\int_{\mathbb{R}^m}\left\{D(y) + \frac{1}{\varepsilon}b(y)\otimes b(y)\right\} : \nabla_y u^\varepsilon(s,y)\otimes\nabla_y u^\varepsilon(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad= \frac{1}{2}\int_{\mathbb{R}^m}\big(u^\varepsilon_{\text{in}}(y)\big)^2\,\mathrm{d}y. \end{aligned}$$

In particular, when $\varepsilon \searrow 0$, $(u^\varepsilon)_\varepsilon$ converges, at least weakly $\star$ in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ towards some limit $u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$. Notice that the explicit methods are not well adapted for the numerical approximation of (1), (2) when $\varepsilon \searrow 0$, since the CFL condition leads to severe time step constraints like

$$\frac{d}{\varepsilon}\frac{\Delta t}{|\Delta y|^2} \leqslant \frac{1}{2}$$

where Δt is the time step and Δy is the grid spacing. In such cases implicit methods are desirable [2,21]. For the numerical resolution of diffusion equations on distorted grids we refer to [17,16,20]. Finite volume methods have been discussed in [14,1]. Recent results concerning anisotropic elliptic problems and non-linear heat equations were obtained in [11,12,15].

In plasma physics, the collision operator gives rise to anisotropic diffusion in velocity space due to the interaction between particles and waves [22]. The applications we have in mind concern the magnetic confinement. This analysis is required when studying the energy (temperature)