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A note on semilinear heat equation in hyperbolic space ☆

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Abstract

Bandle et al. [1] obtained a quite interesting result about a semilinear heat equation that the Fujita exponent relative to the whole hyperbolic space is just the same as that relative to bounded domain in Euclidean space, and, in addition, the properties of solutions are different in the critical exponent case. Our purpose is to answer an open problem proposed by Bandle et al. for the critical exponent case, and it, together with the one obtained by them, shows that the critical exponent case does belong to the non-blow-up case, which is completely different from the case in Euclidean space.

MSC: 35B33; 35K05; 35K15

Keywords: Semilinear heat equations; Hyperbolic space; Blow-up; Fujita exponent

1. Introduction

In this paper, we are concerned with the Cauchy problem of a semilinear heat equation in the hyperbolic space

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{\mathbb{H}^N} u + e^{\alpha t}|u|^{p-1}u, & (x,t) \in \mathbb{H}^N \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{H}^N, \end{cases} \tag{1.1}$$

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where $\mathbb{H}^N$ is the N dimensional hyperbolic space, $\Delta_{\mathbb{H}^N}$ is the Laplace–Beltrami operator on $\mathbb{H}^N$, α and p are constants with $\alpha > 0$ and $p > 1$, and $u_0(x)$ is a positive function in $C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$.

It were Bandle et al. [1] who recently obtained a quite interesting result that the solutions of the Cauchy problem (1.1) behave like the solutions of the corresponding problem in Euclidean space for bounded domain rather than the whole space, see also [7]. To our knowledge, this is the first study on manifolds with negative curvatures. For relative works in Euclidean space $\mathbb{R}^N$ and general manifolds with nonnegative Ricci curvatures, see the reviews [5,3] and Zhang's work [9]. Let $p_H^* := 1 + \frac{\alpha}{\lambda_0}$ with $\lambda_0 = \frac{(N-1)^2}{4}$. Bandle et al. have shown that p_H^* is the Fujita exponent of the problem (1.1). Precisely speaking, they proved that if $1 < p < p_H^*$, then every nontrivial solution blows up in finite time; while if $p > p_H^*$, then the problem (1.1) possesses global solutions for small initial data. As for the critical exponent case $p = p_H^*$, they proved that if $\alpha > \frac{2}{3}\lambda_0$, then there exist global solutions. The purpose of this note is to show that this phenomenon is still valid even for small α. More precisely, we shall prove the following theorem.

Theorem 1. *For the critical exponent case $p = p_H^*$, if $0 < \alpha \leqslant \frac{2}{3}\lambda_0$, then there exists a positive global solution of the problem (1.1) with initial datum $u_0 \in C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$ with $u_0(x) > 0$.*

Remark 1. Noticing that $\lambda_0 = \frac{(N-1)^2}{4}$ is the infimum of the L^2 spectrum of $-\Delta_{\mathbb{H}^N}$, see [2], the critical exponent in [1] is similar to the one in [7, Theorem 4] for the initial–boundary value problem with homogeneous Dirichlet boundary conditions on the bounded domain in $\mathbb{R}^N$, where Meier proved that the critical exponent is $p_H^* := 1 + \frac{\alpha}{\lambda_1}$ with λ_1 being the first Dirichlet eigenvalue of $-\Delta$ in D.

Remark 2. For the critical exponent case, Bandle et al.'s results and Theorem 1 reveal a new interesting phenomenon: in the hyperbolic space the critical exponent is *not* a blow-up exponent, which is completely different from the same problem posted in bounded domain of $\mathbb{R}^N$, which will be presented in Appendix A.[1]

2. Proof of Theorem 1

The key ingredient of our proof is a transformation, which translates the problem (1.1) to a parabolic equation that does not explicitly contain the time t. To be more precise, suppose that $u(x,t)$ is a solution of (1.1). Let

$$v(x,t) = e^{\lambda t} u(x,t), \tag{2.1}$$

where $\lambda = \frac{\alpha}{p-1}$. Then $v(x,t)$ satisfies

$$\begin{cases} v_t = \Delta_{\mathbb{H}^N} v + \lambda v + |v|^{p-1} v, & (x,t) \in \mathbb{H}^N \times (0,T), \\ v(x,0) = u_0(x), & x \in \mathbb{H}^N. \end{cases} \tag{2.2}$$

[1] Meier [7] did not write down the proof of that the critical exponent belongs to the blow-up case.