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Long-time asymptotics for the porous medium equation: The spectrum of the linearized operator ☆

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Abstract

We compute the complete spectrum of the displacement Hessian operator, which is obtained from the confined porous medium equation by linearization around its stationary attractor, the Barenblatt profile. On a formal level, the operator is conjugate to the Hessian of the entropy via similarity transformation. We show that the displacement Hessian can be understood as a self-adjoint operator and find that its spectrum is purely discrete. The knowledge of the complete spectrum and the explicit information about the corresponding eigenfunctions give new insights on the convergence and higher order asymptotics of solutions to the porous medium equation towards its attractor. More precisely, the inspection of the eigenfunctions allows to identify symmetries in $\mathbb{R}^N$ with flows whose rates of convergence are faster than the uniform, translation-governed bound. The present work complements the analogous study of Denzler & McCann for the fast-diffusion equation.
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1. Introduction

In this paper, we study the long-time asymptotics of nonnegative solutions to the porous medium equation, i.e.,

$$\partial_t \rho - \Delta\left(\rho^m\right) = 0 \quad \text{in } \mathbb{R}^N, \tag{1}$$

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with exponent $m > 1$. This equation is best known for modeling the flow of an isentropic gas through a porous medium; other applications include groundwater infiltration, population dynamics, and heat radiation in plasmas (cf. [32, Chapter 2]).

Solutions to (1) feature different phenomena depending on the degree of the nonlinearity ρ^m. In the case $m = 1$, Eq. (1) is the ordinary diffusion (or heat) equation. For $0 < m < 1$, the diffusion flux $m\rho^{m-1}$ diverges as ρ vanishes and thus, for suitable initial data, the solution spreads over the whole $\mathbb{R}^N$ immediately. In this situation, Eq. (1) is often called the *fast diffusion* equation. In the porous-medium range $m > 1$, the diffusion flux increases with the density and degenerates where $\rho = 0$. Consequently, solutions will preserve a compact support and hence this type of propagation goes by the name *slow diffusion*. In the present paper, we restrict our attention exclusively to the latter case. For a study of the fast diffusion equation and more general evolutions of porous-medium type, we refer to VÁZQUEZ [31] and references therein.

The Cauchy problem for the porous medium equation is solved in various settings. As the equation is degenerate parabolic, that is, it is parabolic only where the solution is positive, solutions are in general not classical. More precisely, if the initial datum is zero in some open subset of $\mathbb{R}^N$, then there is a slowly propagating free boundary that separates the region where the solution is positive from the region where it is zero. For suitable initial configurations, e.g. $0 \leqslant \rho_0 \in L^1(\mathbb{R}^N)$, unique strong solutions are known to exist, and these solutions are bounded and continuous. Moreover, strong solutions preserve total mass, $\|\rho(t,\cdot)\|_{L^1(\mathbb{R}^N)} = \|\rho_0\|_{L^1(\mathbb{R}^N)}$ for all $t > 0$. The Cauchy problem for the porous medium equation is reviewed by VÁZQUEZ in [32].

It is well-known [36,3,24] that the porous medium equation allows for self-similar solutions, so-called Barenblatt solutions, propagating on the scale $|x| = t^\alpha$ where

$$\alpha = \frac{1}{N(m-1)+2},$$

and given by

$$\rho_*(t,x) = \frac{1}{t^{N\alpha}}\left(L - \frac{\alpha(m-1)}{2m}\frac{|x|^2}{t^{2\alpha}}\right)_+^{1/(m-1)}. \tag{2}$$

Here, $(\cdot)_+ = \max\{\cdot, 0\}$ and L is an arbitrary constant that can be fixed, for instance, by normalizing the total mass or by choosing the radius of the support of ρ_*. It is interesting to note that although the Barenblatt solution is a strong solution of the porous medium equation, it is not a solution of the Cauchy problem as ρ_* diverges to the delta measure δ_0 (times a constant depending on L) at time zero — a reason for which it is often called a "source-type solution".

The Barenblatt solution describes the long-time behavior of any solution ρ having same total mass as ρ_*. Indeed, for arbitrary nonnegative initial data in L^1, solutions spread with a finite propagation speed over the whole space and, while diffusing, the shape of the solution becomes asymptotically close to the profile of the Barenblatt solution,

$$\rho(t,\cdot) \approx \rho_*(t,\cdot) \quad \text{as } t \gg 1.$$

This long-time behavior was intensively studied over many years, starting with the work by KAMIN [15,16], who established uniform convergence in one space dimension. The generalization to several dimensions is due to FRIEDMAN & KAMIN [12] and KAMIN & VÁZQUEZ [17].

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