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A unified proof on the partial regularity for suitable weak solutions of non-stationary and stationary Navier–Stokes equations

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Abstract

The partial regularity of the suitable weak solutions to the Navier–Stokes equations in $\mathbb{R}^n$ with $n = 2, 3, 4$ and the stationary Navier–Stokes equations in $\mathbb{R}^n$ for $n = 2, 3, 4, 5, 6$ are investigated in this paper. Using some elementary observation of these equations together with De Giorgi iteration method, we present a unified proof on the results of Caffarelli, Kohn and Nirenberg [1], Struwe [17], Dong and Du [5], and Dong and Strain [7]. Particularly, we obtain the partial regularity of the suitable weak solutions to the 4d non-stationary Navier–Stokes equations, which improves the previous result of [5], where Dong and Du studied the partial regularity of smooth solutions of the 4d Navier–Stokes equations at the first blow-up time.

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1. Introduction

The incompressible non-stationary and stationary Navier–Stokes equations can be written as follows

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \qquad \operatorname{div} u = 0, \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$-\Delta u + u \cdot \nabla u + \nabla p = f, \qquad \operatorname{div} u = 0, \quad x \in \Omega, \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded regular domain for $n = 2, 3, 4$ in (1.1) and $n = 2, 3, 4, 5, 6$ in (1.2). Here u describes velocity vector field of the flow, the boundary values of u are assumed to be zero, the scalar function p stands for the pressure of the fluid and the external force is denoted by f with $\operatorname{div} f = 0$. The initial data of (1.1) u_0 satisfies $\operatorname{div} u_0 = 0$.

The mathematical theory of the Navier–Stokes equations extends at least as far back as Leray in 1930s. The global in time existence of the weak solutions to (1.1) is proved when the spatial dimension is two or three. Hopf also obtained that there exists at least one weak solution to (1.1) on a bounded domain in $\mathbb{R}^3$.

Theorem *(Leray–Hopf). Let $u_0 \in L^2(\Omega)$ be a divergence-free vector field and $f \in L^2(0,T; L^2(\Omega))$. Then there exists a function u such that*

(1) $u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega))$,
(2) (u,p) *solves* (1.1) *in* $\Omega \times (0,T)$ *in the sense of distributions,*
(3) (u,p) *satisfies the energy inequality for* $t < T$

$$\frac{1}{2}\int_\Omega |u(x,t)|^2\,dx + \int_0^t \int_\Omega |\nabla u(x,\tau)|^2\,dx\,d\tau \leqslant \frac{1}{2}\int_\Omega |u_0(x)|^2\,dx + \int_0^t\int_\Omega |f(x,\tau)|^2\,dx\,d\tau. \tag{1.3}$$

The weak solution of 2d Navier–Stokes equations is regular, which is proved by Leray. However, it is still an open problem to show that the weak solution of Navier–Stokes equations in $\mathbb{R}^3$ is classical solution. For an introduction and extensive study of the stationary Navier–Stokes equations (1.2), the reader can refer to monograph [10].

It is well known that there exists another kind of weak solutions to the Navier–Stokes equations, that is, suitable weak solution originated from Scheffer [14]. Scheffer studied the Hausdorff measure of possible singular points set of the solutions to the 3d and 4d Navier–Stokes equations in series of papers [13–15]. Later, Caffarelli, Kohn and Nirenberg improved Scheffer's result in three dimension in a celebrated paper [1], where they showed that one-dimensional parabolic Hausdorff measure of singular points for any suitable weak solution to the Navier–Stokes equations is zero. From then on, the investigation of partial regularity of Navier–Stokes equations attracts more and more attention [5,7,11,12,17,18]. More precisely, new proof on the partial regularity of 3d Navier–Stokes equations is given. Based on a blow-up procedure and compact method, Lin [12] presented a simple proof on the theorem of Caffarelli, Kohn and Nirenberg with