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# On the concentration of semi-classical states for a nonlinear Dirac–Klein–Gordon system

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**Abstract**

In the present paper, we study the semi-classical approximation of a Yukawa-coupled massive Dirac–Klein–Gordon system with some general nonlinear self-coupling. We prove that for a constrained coupling constant there exists a family of ground states of the semi-classical problem, for all $\hbar$ small, and show that the family concentrates around the maxima of the nonlinear potential as $\hbar \to 0$. Our method is variational and relies upon a delicate cutting off technique. It allows us to overcome the lack of convexity of the nonlinearities.


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*Keywords:* Dirac–Klein–Gordon system; Semi-classical states; Concentration

## 1. Introduction and main result

In this paper we study the solitary wave solutions of the massive Dirac–Klein–Gordon system involving an external self-coupling:

$$\begin{cases} i\dfrac{\hbar}{c}\partial_t\psi + i\hbar\displaystyle\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc\beta\psi - \lambda\phi\beta\psi = f(x,\psi), \\ \dfrac{\hbar^2}{c^2}\partial_t^2\phi - \hbar^2\Delta\phi + M\phi = 4\pi\lambda(\beta\psi)\cdot\psi \end{cases} \tag{1.1}$$

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for $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, where $c$ is the speed of light, $\hbar$ is Planck's constant, $\lambda > 0$ is coupling constant, $m$ is the mass of the electron and $M$ is the mass of the meson (we use the notation $u \cdot v$ to express the inner product of $u, v \in \mathbb{C}^4$). Here $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\beta$ are $4 \times 4$ complex Pauli matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1,2,3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

System (1.1) arises in mathematical models of particle physics, especially in nonlinear topics. Physically, system (1.1) describes the Dirac and Klein–Gordon equations coupled through the Yukawa interaction between a Dirac field $\psi \in \mathbb{C}^4$ and a scalar field $\phi \in \mathbb{R}$ (see [6]). This system is inspired by approximate descriptions of the external force involve only functions of fields. The nonlinear self-coupling $f(x,\psi)$, which describes a self-interaction in Quantum electrodynamics, gives a closer description of many particles found in the real world. Various nonlinearities are considered to be possible basis models for unified field theories (see [20,21,23], etc. and references therein).

System (1.1) with null external self-coupling, i.e., $f \equiv 0$, has been studied for a long time and results are available concerning the Cauchy problem (see [7–9,25,28], etc.). The first result on the global existence and uniqueness of solutions of (1.1) (in one space dimension) was obtained by J.M. Chadam in [8] under suitable assumptions on the initial data. For later developments, we mention, e.g., that J.M. Chadam and Robert T. Glassey [9] yield the existence of a global solution in three space dimensions. In [7], N. Bournaveas obtained low regularity solutions of the Dirac–Klein–Gordon system by using classical Strichartz-type time–space estimates.

As far as the existence of stationary solutions (solitary wave solutions) of (1.1) is concerned, there is a pioneering work by M.J. Esteban, V. Georgiev and E. Séré (see [19]) in which a multiplicity result is studied. Here, by stationary solution, we mean a solution of the type

$$\begin{cases} \psi(t,x) = \varphi(x)e^{-i\xi t/\hbar}, \quad \xi \in \mathbb{R}, \quad \varphi : \mathbb{R}^3 \to \mathbb{C}^4, \\ \phi = \phi(x). \end{cases} \tag{1.2}$$

In [19], using the variational arguments, the authors obtained infinite many solutions for $\xi \in (-\frac{mc}{\hbar}, 0)$ under the assumption

$$\varphi(x) = \begin{pmatrix} v(r) & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iu(r) & \begin{pmatrix} \cos\vartheta \\ e^{i\tau}\sin\vartheta \end{pmatrix} \end{pmatrix}$$

where $(r, \vartheta, \tau)$ are the spherical coordinates of $x \in \mathbb{R}^3$.

We emphasize that the works mentioned above were mainly concerned with the autonomous system with null self-coupling. Besides, limited work has been done in the semi-classical approximation. For small $\hbar$, the solitary waves are referred to as semi-classical states. To describe the transition from quantum to classical mechanics, the existence of solutions $(\varphi_\hbar, \phi_\hbar)$, $\hbar$ small, possesses an important physical interest. In the present paper we are devoted to the existence