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Global conservative solutions for a model equation for shallow water waves of moderate amplitude [☆]

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Abstract

In this paper, we study the continuation of solutions to an equation for surface water waves of moderate amplitude in the shallow water regime beyond wave breaking (in [11], Constantin and Lannes proved that this equation accommodates wave breaking phenomena). Our approach is based on a method proposed by Bressan and Constantin [2]. By introducing a new set of independent and dependent variables, which resolve all singularities due to possible wave breaking, the evolution problem is rewritten as a semilinear system. Local existence of the semilinear system is obtained as fixed points of a contractive transformation. Moreover, this formulation allows one to continue the solution after collision time, giving a global conservative solution where the energy is conserved for almost all times. Finally, returning to the original variables, we obtain a semigroup of global conservative solutions, which depend continuously on the initial data.

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Keywords: Shallow water waves of moderate amplitude; Conservative solutions; Wave breaking

1. Introduction

This paper is focused on an equation for surface waves of moderate amplitude in the shallow water regime

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$$\begin{cases} u_t + u_x + 6uu_x - 6u^2u_x + 12u^3u_x + u_{xxx} \\ \quad - u_{xxt} + 14uu_{xxx} + 28u_xu_{xx} = 0, \quad t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

which arises as an approximation to the Euler equations, modeling the unidirectional propagation of surface water wavers.

The study of water waves is a fascinating subject because the phenomena are familiar and the mathematical problems are various [29]. Since the exact governing equations for water waves have proven to be nearly intractable, the quest for suitable simplified model equations was initiated at the earliest stages of development of hydrodynamics. Until the early twentieth century, the study of water waves was confined almost exclusively to linear theory. Since linearization failed to explain some important aspects, several nonlinear models have been proposed, explaining nonlinear behaviors linking breaking waves and solitary waves [11]. The most prominent example is the Korteweg–de Vries (KdV) equation [24], the only member of the wider family of BBM-type equations that is integrable and relevant for the phenomenon of soliton manifestation [1]. Since KdV and BBM equations do not model breaking waves (wave breaking means that the wave remains bounded but its slope becomes unbounded in finite time), several model equations were proposed to capture this phenomenon, one of the typical models is the Camassa–Holm equation:

$$u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_xu_{xx} + uu_{xx}. \tag{1.2}$$

Eq. (1.2) was first obtained as a bi-Hamiltonian generalization of KdV equation by Fuchssteiner and Fokas [16], and later derived as a model for unidirectional propagation of shallow water over a flat bottom by Camassa and Holm [4]. Similar to the KdV equation, Camassa–Holm equation has also a bi-Hamiltonian structure [16,25] and is completely integrable [4,7,22]. The orbital stability of solitary waves and the stability of the peakons ($k = 0$) for Camassa–Holm equation are considered by Constantin and Strauss [12,13]. The advantage of the Camassa–Holm equation in comparison with the KdV equation lies in the fact that the Camassa–Holm equation has peaked solitons and models the peculiar wave breaking phenomena (cf. [5,8]). Many results have been obtained for waves of small amplitude, but it is also interesting and important to look at large amplitude waves. Departing from an equation derived by Johnson in [23], which at a certain depth below the fluid surface is a Camassa–Holm equation, one can derive a corresponding equation for the free surface valid for waves of moderate amplitude in the shallow water regime. Local well-posedness for initial value problem associated to (1.1) was first established by Constantin and Lannes [11], and then improved using Kato's semigroup approach for quasi-linear equations and an approach due to Kato by Duruk Mutlubas [14]. Recently, Mi and Mu [26] improved the local well-posedness of Eq. (1.1) in Besov space $B^s_{p,r}$ with $1 \leqslant p, r \leqslant +\infty$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ by the transport equations theory and the classical Friedrichs regularization method. Note that, unlike KdV or CH, Eq. (1.1) does not have a bi-Hamiltonian integrable structure [6]. Nevertheless, the equation possesses solitary wave profiles that resemble those of CH, analyzed in [9], and present similarities with the shape of the solitary waves for the governing equations for water waves considered in [10,13,17], and the orbital stability of solitary waves for this equation was recently obtained in [15,17].

In view of the possible development of singularities in finite time, continuation of the solution beyond wave breaking has been a challenge. Recently, this issue has been discussed for the Camassa–Holm equation [2,3] and for the hyperelastic rod equation in [27,28,30], by introducing