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J. Differential Equations 256 (2014) 1873–1902

Journal of Differential Equations

www.elsevier.com/locate/jde

The non-existence results for a class of integral equation ☆

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Received 12 July 2013; revised 27 November 2013

Available online 23 December 2013

Abstract

In this paper, we consider the following integral system

$$u(x,b)=\int\limits_{\mathbb{R}^n}\frac{u^q(y,b)}{(b+|x-y|)^\lambda}\,dy, \tag{0.1}$$

which is related to the weak type convolution-Young's inequality. Under the assumption of that $\lambda \in (0,n)$ and $0<q\leqslant n/\lambda$, we show that system (0.1) doesn't have a positive solution in $L^q_{loc}(\mathbb{R}^n)$. Furthermore, we prove that as $\lambda\in(0,n-1/3)$ and $q=2n/\lambda-1$, system (0.1) does not admit positive solution in $L^{q+1}(\mathbb{R}^n)$ $(n>2)$, which implies that the maximizing pair of the weak type convolution-Young's inequality with kernel function $(b+|x|)^{-\lambda}$ does not exist. Meanwhile, for $\lambda\in(-\infty,0)$ and $q=2n/\lambda-1$, we also show that the system (0.1) doesn't admit non-negative Lebesgue measurable solution. This is distinct from the original conformal invariant integral system.

MSC: 45G15; 45M99

Keywords: Weak type convolution-Young's inequality; Integral equation; Conformal invariance

☆ Supported by the National Natural Science Foundation of China (Grant Nos. 11071200, 11126148, 1127130, 11371295), the Young Foundation of Hunan Agriculture University (No. 12QN08) and the Project Supported by Scientific Research Fund of Hunan Provincial Education Department (No. 13C395).

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.12.009

1. Introduction

In this paper, we study the non-existence of positive solutions for the following integral equation

$$u(x,b)=\int_{\mathbb{R}^n}\frac{u^q(y,b)}{(b+|x-y|)^\lambda}\,dy. \tag{1.1}$$

The motivation for studying this system comes from the maximizing pair (f,g) and its related sharp constant of the weak type convolution-Young's inequality (see (1.4) below). To figure out the origin of this system, we firstly introduce some necessary notions and background. Let $\mathbb{R}^n$, $n\geqslant 2$, be the n dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with normalized Lebesgue measure $d\mu=d\mu(\cdot)$. For a measurable function f on $\mathbb{R}^n$, let us recall the distribution of f defined by

$$f_*(t)=\lambda_f(t)=\mu\big(\big\{x\in\mathbb{R}^n\colon\ \big|f(x)\big|>t\big\}\big).$$

For $0<p<\infty$, the $L^p(\mathbb{R}^n)$ and $L^{p,\infty}(\mathbb{R}^n)$ will denote the set of all measurable function on $\mathbb{R}^n$ such that

$$L^p\big(\mathbb{R}^n\big)=\left\{f\colon\ \|f\|_{L^p(\mathbb{R}^n)}=\left(\int_{\mathbb{R}^n}\big|f(x)\big|^p\,dx\right)^{1/p}=\left(p\int_0^\infty t^{p-1}\lambda_f(t)\,dt\right)^{1/p}<\infty\right\}, \tag{1.2}$$

and

$$L^{p,\infty}\big(\mathbb{R}^n\big)=\left\{f\colon\ \|f\|_{L^{p,\infty}(\mathbb{R}^n)}=\left(\frac{n}{\mu(\mathbb{S}^{n-1})}\right)^{1/p}\sup_{\gamma>0}\gamma\big(\lambda_f(\gamma)\big)^{1/p}\right\}. \tag{1.3}$$

For simplicity, we denote $\|\cdot\|_{L^p(\mathbb{R}^n)}$ and $\|\cdot\|_{L^{p,\infty}(\mathbb{R}^n)}$ by $\|\cdot\|_p$ and $\|\cdot\|_{p,\infty}$, respectively. For every positive function $h\in L^{p,\infty}(\mathbb{R}^n)$, the weak type convolution-Young's inequality states that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(x)h(x-y)g(y)\,dx\,dy\leqslant C(p,s,r)\|h\|_{p,\infty}\|f\|_r\|g\|_s \tag{1.4}$$

for all $f\in L^r(\mathbb{R}^n)$, $g\in L^s(\mathbb{R}^n)$, $1<s,p,r<\infty$ and $1/s+1/p+1/r=2$.

To prove the existence of the sharp maximizing pair (f,g) in (1.4) and explicitly compute the best constant $C(s,n,r)$ and (f,g), we maximize the functional:

$$J(f,g)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(x)g(y)h(x-y)\,dx\,dy, \tag{1.5}$$

under the constraint conditions:

$$\|f\|_r=\|g\|_s=1,\quad\text{and}\quad f,g>0.$$