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Cyclicity of a simple focus via the vanishing multiplicity of inverse integrating factors ☆

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ABSTRACT

First we provide new properties about the vanishing multiplicity of the inverse integrating factor of a planar analytic differential system at a focus. After we use this vanishing multiplicity for studying the cyclicity of foci with pure imaginary eigenvalues and with homogeneous nonlinearities of arbitrary degree having either its radial or angular speed independent of the angle variable in polar coordinates. After we study the cyclicity of a class of nilpotent foci in their analytic normal form.

1. Introduction and statement of the results

We consider planar differential systems

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

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where $P, Q : U \to \mathbb{R}$ are $\mathcal{C}^1$ functions defined in the simple connected open subset U of $\mathbb{R}^2$. A $\mathcal{C}^1$ function $R : U \to \mathbb{R}$ such that

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y} \tag{2}$$

is an *integrating factor* of system (1). The differential systems (1) having an integrating factor in U have a first integral $H : U \to \mathbb{R}$ satisfying that

$$RP = \frac{\partial H}{\partial y}, \qquad RQ = -\frac{\partial H}{\partial x}.$$

As usual a *first integral* $H : U \to \mathbb{R}$ is a function constant on the solutions of the differential system (1).

It is immediate to check that R is an integrating factor of system (1) in U if and only if R is a solution of the linear partial differential equation

$$P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)R \tag{3}$$

in U.

A $\mathcal{C}^1$ function $V : U \to \mathbb{R}$ is an *inverse integrating factor* if V verifies the linear partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V \tag{4}$$

in U. We note that V satisfies (4) in U if and only if $R = 1/V$ satisfies (3) in $U \setminus \Sigma$ where $\Sigma = \{(x, y) \in U\colon\ V(x, y) = 0\}$.

In 1996 it was proved in [11] the following result. Assume that the $\mathcal{C}^1$ planar differential system (1) defined in the open subset U of $\mathbb{R}^2$ has an inverse integrating factor $V : U \to \mathbb{R}$. If γ is a limit cycle of system (1) contained in U, then γ is contained in Σ. For an easier proof see [13]. After this result many papers have been published studying different aspects of the limit cycles using the properties of the inverse integrating factor. For a good survey see [8].

First in this paper we provide some new properties on the vanishing multiplicity of the inverse integrating factor of a planar analytic differential system, see Proposition 1. Later on we use this vanishing multiplicity for studying the cyclicity of some foci of several classes of planar polynomial differential systems.

We deal with real planar analytic differential system with a *monodromic singular point* at the origin, i.e. we consider differential systems (1) where $P(x, y)$ and $Q(x, y)$ are real analytic functions in a neighborhood $\mathcal{U}$ of the origin such that $P(0, 0) = Q(0, 0) = 0$, and the origin is either a focus or a center. A *focus* is a singular point such that in a neighborhood of it all the orbits different from the singular point spiral either tending to it or going away from it. A *center* is a singular point having a neighborhood filled of periodic orbits with the unique exception of the singular point.

We will only consider analytic system (1) being the origin a *simple focus*, i.e. the monodromic singular point is one of the following three types: non-degenerate focus, degenerate focus without characteristic directions or nilpotent focus (see the definitions in Section 2). System (1) having a simple monodromic singular point, after performing a *generalized* polar blow-up, can be transformed into a differential equation defined over a cylinder blowing up the origin into a periodic orbit. More precisely, performing a generalized polar blow-up, system (1) defined in a neighborhood $\mathcal{U}$ of the origin passes to be defined into a cylinder $C = \{(r, \theta) \in \mathbb{R} \times \mathbb{S}^1\colon\ |r| < \delta\}$ for a certain $\delta > 0$ sufficiently small. Here, we have considered the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}T$ where $T > 0$ is the constant period associated to the polar change and $\mathbb{Z}T = \{kT\colon\ k \in \mathbb{Z}\}$. This change to polar coordinates is a diffeomorphism in $\mathcal{U}\backslash\{(0, 0)\}$ and transforms the origin of coordinates into the circle $r = 0$. In fact, the neighborhood $\mathcal{U}$