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Regularization by $\frac{1}{2}$-Laplacian and vanishing viscosity approximation of HJB equations

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ABSTRACT

We investigate the regularizing effect of adding small fractional Laplacian, with critical fractional exponent $\frac{1}{2}$, to a general first order HJB equation. Our results include some regularity estimates for the viscosity solutions of such perturbations, making the solutions classically well-defined. Most importantly, we use these regularity estimates to study the vanishing viscosity approximation to first order HJB equations by $\frac{1}{2}$-Laplacian and derive an explicit rate convergence for the vanishing viscosity limit.

1. Introduction

Within the field of fully nonlinear partial differential equations, the Hamilton–Jacobi–Bellman-type equations are one of the most widely studied class. Among others, the notion of viscosity solutions has been of immense help to achieve a deeper understanding of fully nonlinear PDEs. It is well documented in the literature that the regularizing effect of adding small diffusion to first order fully nonlinear HJB equations has played a pivotal role in the development and understanding of viscosity

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solution theory. In this article we also set out to study a similar problem by adding a small fractional diffusion to a class of fully nonlinear first order HJB equations and investigate the regularizing effect and convergence properties of such approximations. We are interested in the following initial value problem

$$\begin{cases} u_t + H\big(t, x, u(t,x), \nabla u(t,x)\big) = 0 & \text{if } (t,x) \in (0,T] \times \mathbb{R}^n, \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

and its vanishing viscosity approximation

$$\begin{cases} u_t^\epsilon + H\big(t, x, u^\epsilon(t,x), \nabla u^\epsilon(t,x)\big) + \epsilon(-\Delta)^{\frac{s}{2}} u^\epsilon = 0 & \text{if } (t,x) \in (0,T] \times \mathbb{R}^n, \\ u(0,x) = u_0(x). \end{cases} \tag{1.2}$$

In the above T is a positive constant, the Hamiltonian H is a real valued function on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $\epsilon > 0$ is a small positive number. The precise structural assumptions on H will be detailed in Section 2, but roughly speaking, it is a Lipschitz continuous function in all its variables and enjoys some monotonicity property in u. The initial data $u_0(x)$ is a Lipschitz continuous function on $\mathbb{R}^n$. The number $\frac{s}{2}$ in (1.2) is the fractional power of the diffusion operator and s is supposed to be ranging within $[1,2]$.

Among others, a rich source for equations of type (1.1) is the area of optimal control. The value function of a controlled dynamical system or that of a differential game solves an equation of the form (1.1). Also, the equations of type (1.2) are of paramount importance due to their appearance in optimal control of stochastic dynamical systems with α-stable noise. The problem (1.2) is clearly a perturbation of (1.1). From the optimal control viewpoint, if the controlled deterministic dynamical system is perturbed by a small additive Lévy noise then the resulting value function of the perturbed control problem would satisfy an equation of type (1.2). Our aim in this article is to study the stability of such perturbation and its regularizing effect on the value function.

For $s = 2$, the problem (1.2) becomes the classical parabolic approximation of (1.1) and the classical theory for semilinear parabolic equations applies. As a result, for $s = 2$, the Cauchy problem (1.2) is well-posed and the solution u^ϵ is smooth (cf. [6]). It is also well known that the sequence of functions $(u^\epsilon)_{\epsilon>0}$ converges locally uniformly to a function u as $\epsilon \downarrow 0$, which is characterized as the unique *viscosity solution* of (1.1). There are a number of methods available (cf. [1,9]) to estimate the rate of convergence, and one can optimally estimate the error to be of the order $\epsilon^{\frac{1}{2}}$.

The case $s < 2$ is much less classical. The operator $(-\Delta)^{s/2}$ has the following representation (cf. [11]):

$$(-\Delta)^{\frac{s}{2}} u(x) = C(n,s) \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+s}}\, dy. \tag{1.3}$$

The constant $C(n,s)$ depends only on n and s. The above integral in (1.3) should be understood in the principal value sense. Clearly, in view of (1.3), the problem (1.2) is nonlocal in nature or, in other words, an integro-partial differential equation. However, the notion of viscosity solution does make sense for such equations and the literature addressing this notion and related issues is fairly well developed by now. We refer to the articles [2,3,7,12,9,13] and the references therein for more on this topic. The issues addressed in these papers range from standard well-posedness theory to more subtle questions related to regularity.

For $1 < s < 2$, the question on regularization was first answered by C. Imbert [7]. It was shown, under certain conditions, that the unique viscosity solution of (1.2) is indeed of class $C^{1,2}$. In other words, the perturbed equation (1.2) is classically well-defined and the perturbation has the same effect as classical parabolic regularization. In [7], the author also gives a condition on the Hamiltonian H under which u^ϵ becomes C^∞. The L^∞-error bound on $u^\epsilon - u$ is estimated to be of the order $\epsilon^{\frac{1}{s}}$. The