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On the existence and non-existence of bounded solutions for a fourth order ODE

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ABSTRACT

We prove that if $F \in C^1(\mathbb{R})$ is coercive and $\{F' = 0\}$ is discrete, then the EFK equation

$$u'''' - c^2 u'' + F'(u) = 0 \tag{1}$$

possesses $L^\infty(\mathbb{R})$ solutions if and only if F' changes sign at least twice. As a corollary we prove that if u_n solves

$$u_n'''' + c_n^2 u_n'' + F'(u_n) = 0,$$

then $\|u_n\|_\infty \to +\infty$ if $c_n \to 0$, provided F has a unique local minimum, its only minimum is nondegenerate and $\operatorname{int}(\{F' = 0\}) = \emptyset$. Finally we give criteria ensuring existence and non-existence of T-periodic solutions to (1) when F has multiple wells.

1. Introduction

In this paper, we consider a question raised by Lazer and McKenna in [4]. In [5,6], the equation

$$u'''' + c^2 u'' + (1 + u)^+ - 1 = 0 \tag{1.1}$$

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was introduced as a model for traveling waves in suspension bridges. Eq. (1.1) can be also interpreted as a Swift–Hohenberg equation with a non-smooth, single-well nonlinearity. As such, it has a wide variety of applications in pattern formation (see, e.g., the book [9] and the references therein). One property of solutions to (1.1) that soon became apparent from simulations, is that the bounded ones are wildly oscillating when $c \to 0$, in the sense that they must go to $+\infty$ in the L^∞-norm. Therefore it is natural to expect that no bounded solutions to (1.1) can exist for $c = 0$. These two facts were proven in [4], where the authors furthermore asked "*whether the same conclusions can be reached for the more interesting nonlinearity $e^u - 1$ [...] Mathematically, we would also like a more general class of nonlinearity*". The nonlinearity $e^u - 1$ is indeed the one employed by engineers instead of the non-smooth one $(1+u)^+ - 1$ and the corresponding equation is usually known as the *smooth* suspended bridge equation. Regarding the behavior as $c \to 0$, an analogous blow-up result has been proved for the class of homoclinic solutions in [3] for, roughly speaking, single-well potentials which are nondegenerate at the minimum point. This is done proving an explicit lower bound on the L^∞-norm of the solutions, which blows up as $c \to 0$ in (1.1). Regarding the non-existence result for $c = 0$, the case where the nonlinearity is a generic cubic polynomial in u was already considered in [8].

In this paper, we try to answer Lazer and McKenna's questions keeping the nonlinearity as general as possible. As it turns out, the existence of bounded solution to

$$u'''' + F'(u) = 0 \tag{1.2}$$

with F being a smooth coercive potentials with discrete set of critical points, is actually *equivalent* to F having at least two local minima. Moreover, this property extends to the more general equation

$$u'''' - pu'' + F'(u) = 0, \tag{1.3}$$

for nonnegative values of p. In the case $p \geqslant 0$, (1.3) is called the *Extended Fisher–Kolmogorov* (in short, EFK) equation, and its dynamics is very different from the Swift–Hohenberg case $p = -c^2 < 0$ of Eq. (1.1). We obtain the following result (see Theorems 3.1, 5.1).

Theorem 1.1. *Let $F \in C^1(\mathbb{R})$ with $\{F' = 0\}$ being discrete and $\underline{\lim}_{t \to \pm\infty} F(t) = +\infty$. Then, the following alternative holds*:

1. *either F has a unique global minimum and no other local extrema, in which case Eq. (1.3) for $p \geqslant 0$ has no bounded nonconstant solution on $\mathbb{R}$,*
2. *or F has at least two local extrema, in which case (1.3) for $p \geqslant 0$ has a T-periodic nonconstant solution for any sufficiently large T.*

As a consequence of the non-existence result, we extend the results of [3] to bounded solutions of (1.3).

Theorem 1.2. *Let $F \in C^2(\mathbb{R})$ be a coercive potential with a unique global minimum t_0 and no other local extrema. If $\operatorname{int}(\{F' = 0\}) = \emptyset$ and $F''(t_0) > 0$ then, for any family of bounded solutions $\{u_p\}$ of (1.3) with $p < 0$, it holds $\|u_p\|_\infty \to +\infty$ as $p \uparrow 0$.*

Notice that the potential $e^u - u - 1$, corresponding to the nonlinearity $e^u - 1$ in (1.2), satisfies all the hypotheses of the previous two theorems. As a further corollary of Theorem 1.1 we prove the following result, which was well known for the double-well model potential $F(u) = (1-u^2)^2/4$ (see [13, Theorem A]).

Theorem 1.3. *Let $F \in C^1(\mathbb{R})$ be a coercive potential with $\{F' = 0\}$ finite. Then, if T is sufficiently small, there exists no nonconstant periodic solution to (1.3) for $p \geqslant 0$.*