J. Differential Equations 255 (2013) 4244–4266

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Attractivity, multistability, and bifurcation in delayed Hopfield's model with non-monotonic feedback

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ARTICLE INFO

Article history:
Received 30 April 2013
Available online 27 August 2013

MSC:
34K18
34K20
37C70
92B20

Keywords:
Functional differential equations
Neural networks
Global attractivity
Multistability
Bifurcation

ABSTRACT

For a system of delayed neural networks of Hopfield type, we deal with the study of global attractivity, multistability, and bifurcations. In general, we do not assume monotonicity conditions in the activation functions. For some architectures of the network and for some families of activation functions, we get optimal results on global attractivity. Our approach relies on a link between a system of functional differential equations and a finite-dimensional discrete dynamical system. For it, we introduce the notion of strong attractor for a discrete dynamical system, which is more restrictive than the usual concept of attractor when the dimension of the system is higher than one. Our principal result shows that a strong attractor of a discrete map gives a globally attractive equilibrium of a corresponding system of delay differential equations. Our abstract setting is not limited to applications in systems of neural networks; we illustrate its use in an equation with distributed delay motivated by biological models. We also obtain some results for neural systems with variable coefficients.

1. Introduction

In the last decades, there has been an increasing interest in the theory of artificial networks. One of the most famous model is given by the system of delay differential equations

$$u_i'(t) = -u_i(t) + \sum_{j=1}^{s} w_{ij} f_j\big(u_j(t-\tau_{ij})\big), \quad 1 \leqslant i \leqslant s, \tag{1.1}$$

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.08.007

where $u_i(t)$ represents the voltage on the input of neuron i at time t, τ_{ij} denote the synaptic delays and $f_j : \mathbb{R} \to \mathbb{R}$ are the neuron activation functions. The matrix $W = (w_{ij})$ measures the connection strengths between the neurons. If the output from neuron j excites neuron i then $w_{ij} \geqslant 0$ and in the case of inhibitory interaction $w_{ij} \leqslant 0$. System (1.1) without delays was first proposed by Hopfield in [12,13] and later modified by Marcus and Westervelt in [26] incorporating time delays. Such time delays arise from axonal conduction time, distance between the neurons, or finite switching speeds of neurons.

The previous system has been applied in different areas such as classification, associative memory, pattern recognition, parallel computations, and optimization. When a neural circuit is used as an associative memory, stable equilibria correspond with static memory (retrievable) and stable periodic orbits are temporally patterned spike trains. In potential applications of storage and retrieval of information, it is desirable for (1.1) to posses multiple stable patterns (multistability). Unfortunately, most of the network architectures with the usual sigmoid activation functions present a low memory capacity [27]. Motivated by this fact, Morita in [27] and Yoshizawa, Morita, and Amari in [44] have proved that the introduction of a non-monotonic activation function in (1.1) considerably increases its memory capacity. This improvement has deep practical implications in many real situations.

The main goal for system (1.1) is to understand the dynamical behaviors of all trajectories. Due to its complexity, an analysis of bifurcation or stability is still a difficult problem. Moreover, it remains poorly understood how the dynamical behavior depends on the network architecture since the technical difficulties in the study of (1.1) usually impose the restriction to networks of either small size or simple connection architectures. These facts have inspired a large amount of papers; most of them focus on the global convergence to an equilibrium by using Lyapunov functions, the effect of the signal transmission delay, some interesting local phenomena, and multistability by using monotone arguments.

The purpose of this paper is to provide an analytic study of global stability, multistability, and bifurcation in Hopfield's model without assuming monotonicity on f. Our strategy is to link some dynamical behaviors of (1.1) with a discrete system in finite dimension. As we will see, some of our criteria of multistability and global stability are optimal. On the other hand, with our approach we derive criteria of bifurcation without using hard computations of characteristic values or center manifolds. Note that we are dealing with non-monotone systems; comparing with the little progress of the global dynamics in this setting, remarkable developments have been done in the case of monotone networks where generic convergence is guaranteed by the theory of monotone systems [33]. It is also important to recall that the connection of some dynamical behaviors of a discrete equation with some properties of a scalar delay differential equation (DDE) is not new; a systematic approach was initiated by early papers of Mallet-Paret and Nussbaum [24,25], and Ivanov and Sharkovsky [16]; for further generalizations and applications, see, e.g., [11,15,21,22,30,42,43]. The novelty of our method is the connection between systems of DDEs and discrete dynamical systems of dimension higher than one, and the link of the latter with one-dimensional maps. For it, we introduce the notion of strong attractor for a discrete dynamical system, which is more restrictive than the usual concept of attractor when the dimension of the system is higher than one.

This paper is structured as follows. In Section 2, we present the general framework of this paper; we recall some basic notions of discrete systems and show how to derive global convergence of the solutions of (1.1) to an equilibrium from attraction properties of a finite-dimensional map. In Section 3, we apply the results of the previous section to get sufficient conditions for absolute attraction (independent of the size of the delays τ_{ij}) in system (1.1). We also discuss some properties of global bifurcation in (1.1) for a ring of neurons with the activation function considered by Morita in [27]. As a consequence of this discussion, we are able to further describe some steady-state bifurcations studied by Ma and Wu in [23]. Finally, in Section 4, we give an application of our approach to models with distributed delay; we illustrate these results with a functional differential equation recently studied by Yuan and Zhao [45]. We show that it is possible to weaken the conditions for global stability required in the main result of [45].