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Analytic normalization of analytically integrable differential systems near a periodic orbit ☆

Kesheng Wu [a], Xiang Zhang [b,*]

[a] *Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China*

[b] *Department of Mathematics, and MOE–LSC, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China*

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Abstract

For an analytic differential system in $\mathbb{R}^n$ with a periodic orbit, we will prove that if the system is analytically integrable around the periodic orbit, i.e. it has $n-1$ functionally independent analytic first integrals defined in a neighborhood of the periodic orbit, then the system is analytically equivalent to its Poincaré–Dulac type normal form. This result is an extension of analytically integrable differential systems around a singularity to the ones around a periodic orbit.

MSC: 34A34; 34C20; 34C41; 37G05

Keywords: Analytic differential systems; Analytically integrable; Period orbit; Normal form; Analytic normalization

1. Introduction and statement of the main results

Normal form theory has been playing key roles in the study of dynamics for ordinary differential equations and dynamical systems (smooth and discrete ones). Because of its importance, it has been extensively studied, see for instance [1–8,11–13,15,19–24,27,29,30] and the references

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* Corresponding author.
E-mail addresses: kshengwu@gmail.com (K. Wu), xzhang@sjtu.edu.cn (X. Zhang).

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therein. In the normal form theory, one of the main topics is to study the existence of analytic normalization for an analytic dynamical system to its normal form. In this direction there are lots of well known results, which involved the works of Poincaré [16,17], Arnold [2], Bruno [6], Ilyashenko and Yakovenko [12] and so on.

Here we mainly concern the analytically integrable systems. Along this direction Zung [30] showed via torus action that any analytic vector field which is analytically integrable in a neighborhood of the origin in the non-Hamiltonian sense admits a convergent Poincaré–Dulac normalization. Zhang [27] presented a similar result using analytic methods and provided the explicit expression of the normal form, which was not obtained in [30]. Furthermore Zhang [28] extended the results in [27] to degenerate cases and also obtained a result on analytically integrable diffeomorphisms around a fixed point. On the existence of analytic normalization of analytically integrable differential systems, we also refer the readers to [14].

Regarding the integrability and normalization of a differential system near a periodic orbit, Yakovenko [25] studied the existence of C^∞ normalization of a planar C^∞ differential system near a periodic orbit to a very simpler normal form. Peralta-Salas [18] presented a characterization between integrability and normalizers of smooth vector fields in a region filled up with periodic orbits. But to our knowledge it is unsolved whether an analytically integrable differential system around a periodic orbit is analytically equivalent to its Poincaré–Dulac normal form.

Consider the analytic differential system

$$\dot{x} = f(x), \quad x \in \Omega \subset \mathbb{R}^n, \tag{1.1}$$

where the dot denotes the derivative with respect to the time t, Ω is an open subset of $\mathbb{R}^n$ and $f(x) \in C^\omega(\Omega)$. Here $C^\omega(\Omega)$ denotes the ring of analytic functions defined in Ω. Assume that system (1.1) has a periodic orbit, say Γ, located in the region Ω.

We say that system (1.1) is *analytically integrable in a neighborhood of* Γ, if it has $n-1$ functionally independent analytic first integrals defined in the neighborhood of Γ. Here we say that $k > 1$ first integrals defined in $D \subset \Omega$ are *functionally independent* if the gradients of the k first integrals have rank k in a full Lebesgue measure subset of D. A nonconstant function $H(x)$ is a *first integral* of system (1.1) in D if along any orbit located in D of system (1.1) the function H takes a constant value.

Let $x = \varphi(t)$ be an expression of Γ with period T. Since system (1.1) is analytic, the periodic solution $\phi(t)$ is also analytic on $\mathbb{R}$. Taking the transformation $X = x - \varphi(t)$, system (1.1) becomes

$$\dot{X} = f\big(X + \varphi(t)\big) - f\big(\varphi(t)\big). \tag{1.2}$$

It can be written in the form

$$\dot{X} = A(t)X + g(X,t), \quad g(X,t) = O\big(|X|^2\big), \tag{1.3}$$

with $A(t)$ analytic and periodic in t of period T, and $g(X,t)$ analytic in X and t and periodic in t of period T.

By the Floquet theory [10], there is a change of coordinates $X = Q(t)Y$ with $Q(t)$ invertible, analytic and periodic of period T, under which system (1.3) is transformed to

$$\dot{Y} = AY + h(Y,t), \tag{1.4}$$