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J. Differential Equations 256 (2014) 3594–3613

Journal of Differential Equations

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Small global solutions to the damped two-dimensional Boussinesq equations

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Received 7 November 2013; revised 13 February 2014

Available online 7 March 2014

Abstract

The two-dimensional (2D) incompressible Euler equations have been thoroughly investigated and the resolution of the global (in time) existence and uniqueness issue is currently in a satisfactory status. In contrast, the global regularity problem concerning the 2D inviscid Boussinesq equations remains widely open. In an attempt to understand this problem, we examine the damped 2D Boussinesq equations and study how damping affects the regularity of solutions. Since the damping effect is insufficient in overcoming the difficulty due to the "vortex stretching", we seek unique global small solutions and the efforts have been mainly devoted to minimizing the smallness assumption. By positioning the solutions in a suitable functional setting (more precisely, the homogeneous Besov space $\mathring{B}^1_{\infty,1}$), we are able to obtain a unique global solution under a minimal smallness assumption.
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MSC: 35Q35; 35B35; 35B65; 76D03

Keywords: 2D Boussinesq equations; Classical solutions; Global regularity; Small data

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http://dx.doi.org/10.1016/j.jde.2014.02.012
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1. Introduction

This paper examines the global (in time) existence and uniqueness problem on the incompressible 2D Boussinesq equations with damping

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nu u = -\nabla p + \theta \mathbf{e}_2, & x\in\mathbb{R}^2,\ t>0,\\ \partial_t\theta + (u\cdot\nabla)\theta + \lambda\theta = 0, & x\in\mathbb{R}^2,\ t>0,\\ \nabla\cdot u = 0, & x\in\mathbb{R}^2,\ t>0,\\ u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x), & x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where u represents the fluid velocity, p the pressure, $\mathbf{e}_2$ the unit vector in the vertical direction, θ the temperature in thermal convection or the density in geophysical flows, and $\nu>0$ and $\lambda>0$ are real parameters. When νu is replaced by $-\nu\Delta u$ and $\lambda\theta$ by $-\lambda\Delta\theta$, (1.1) becomes the standard viscous Boussinesq equations. (1.1) with $\nu=0$ and $\lambda=0$ reduces to the inviscid 2D Boussinesq equations. If θ is identically zero, (1.1) degenerates to the 2D incompressible Euler equations.

The Boussinesq equations model many geophysical flows such as atmospheric fronts and ocean circulations (see, e.g., [10,16,25,31]). Mathematically the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier–Stokes equations such as the vortex stretching mechanism. The vortex stretching term is the greatest obstacle in dealing with the global regularity issue concerning the Boussinesq equations. When suitable partial dissipation or fractional Laplacian dissipation with sufficiently large index is added, the vortex stretching can be controlled and the global regularity can be established (see, e.g., [1,2,5,6,8,9,11,14,17–24,27, 29,34,35,37]). In contrast, the global regularity problem on the inviscid Boussinesq equations appears to be out of reach in spite of the progress on the local well-posedness and regularity criteria (see, e.g., [7,12,13,15,26,29,30,36]). This work is partially aimed at understanding this difficult problem by examining how damping affects the regularity of the solutions to the Boussinesq equations.

As we know, the issue of global existence and uniqueness relies crucially on whether or not one can obtain global bounds on the solutions. Thanks to the divergence-free condition $\nabla\cdot u=0$, global *a priori* bounds for θ in any Lebesgue space L^q and u in L^2 follow directly from simple energy estimates,

$$\big\|\theta(t)\big\|_{L^q} \leqslant \|\theta_0\|_{L^q}, \qquad \big\|u(t)\big\|_{L^2} \leqslant \|u_0\|_{L^2} + t\|\theta_0\|_{L^2}$$

for $1\leqslant q\leqslant\infty$. However, global bounds for (u,θ) in any Sobolev space, say H^1, cannot be easily achieved and the difficulty comes from the vortex stretching term. More precisely, if we resort to the equations of the vorticity ω and $\nabla^\perp\theta$

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega + \nu\omega = \partial_{x_1}\theta,\\ \partial_t\big(\nabla^\perp\theta\big) + (u\cdot\nabla)\big(\nabla^\perp\theta\big) + \lambda\nabla^\perp\theta = \big(\nabla^\perp\theta\cdot\nabla\big)u, \end{cases} \tag{1.2}$$

we unavoidably have to deal with the "vortex stretching term" $(\nabla^\perp\theta\cdot\nabla)u$, which appears to elude any suitable bound. Here $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. The damping terms are not sufficient to overcome this difficulty. Therefore damping does not appear to make a big difference in dealing with solutions emanating from a general data.

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