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The Poincaré problem, algebraic integrability and dicritical divisors ☆

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Abstract

We solve the Poincaré problem for plane foliations with only one dicritical divisor. Moreover, in this case, we give a simple algorithm that decides whether a foliation has a rational first integral and computes it in the affirmative case. We also provide an algorithm to compute a rational first integral of prefixed genus $g \neq 1$ of any type of plane foliation $\mathcal{F}$. When the number of dicritical divisors $\mathrm{dic}(\mathcal{F})$ is larger than 2, this algorithm depends on suitable families of invariant curves. When $\mathrm{dic}(\mathcal{F}) = 2$, it proves that the degree of the rational first integral can be bounded only in terms of g, the degree of $\mathcal{F}$ and the local analytic type of the dicritical singularities of $\mathcal{F}$.

1. Introduction and results

Denote by $\mathbb{P}^2$ the projective plane over the field of complex numbers. Poincaré, in [36], observed that "to find out whether a differential equation of the first order and of the first degree is algebraically integrable, it is enough to get an upper bound on the degree of the integral.

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Afterwards, one only needs to perform purely algebraic computations". The motivation for this observation, expressed in modern terminology, was the problem of deciding whether a singular algebraic foliation $\mathcal{F}$ on $\mathbb{P}^2$ (plane foliation) has a rational first integral and, when the answer is positive, to compute it. The so-called Poincaré problem consists of obtaining an upper bound of the degree of the first integral depending only on the degree of the foliation. Although it is well-known that such a bound does not exist in general, in the clause (a) of the forthcoming Theorem 1, we shall give a bound of this type under the assumption that the minimal resolution of the singularities of $\mathcal{F}$ (which exists by a result of Seidenberg [38,4]) has only one dicritical (i.e., non-invariant by $\mathcal{F}$) exceptional divisor.

Set $\pi_{\mathcal{F}} : Z_{\mathcal{F}} \to \mathbb{P}^2$ the composition of point blow-ups providing a minimal resolution of the dicritical singularities of $\mathcal{F}$ (see Definition 1). Clause (b) of Theorem 1 states that, under the same assumption and when $\mathcal{F}$ is algebraically integrable, a rational first integral is given by the push-forward by $\pi_{\mathcal{F}}$ of the complete linear system given by a suitable multiple of a specific class in the Picard group of $Z_{\mathcal{F}}$. This gives a very simple procedure, Algorithm 2, to decide, from the resolution $\pi_{\mathcal{F}}$, whether $\mathcal{F}$ has a rational first integral and to compute it in the affirmative case. Alternative algorithms to do this are discussed in Section 4.

The natural extended version of the Poincaré problem consists of bounding the degree of the algebraic integral (reduced and irreducible) invariant curves of a foliation $\mathcal{F}$ (without assuming algebraic integrability) in terms of data obtained from the foliation and/or invariants related to the invariant curves themselves. There has been (and there is) a lot of activity concerning this or related problems, some of the main results (including higher dimension) being [10,8,6,43,40,41, 44,34,16,17,9,18].

The above mentioned problem was stated at the end of the 19th century as the problem of deciding whether a complex polynomial differential equation on the complex plane is algebraically integrable. The usefulness of nonlinear ordinary differential equations in practically any science turns this problem into a very attractive one, especially because when a differential equation admits a first integral, its study can be reduced in one dimension and because it is related to other interesting challenges. For example, it is related to the second part of the XVI Hilbert problem which tries to bound the number of limit cycles for a real polynomial vector field [30,31], with the solutions of Einstein's field equations in general relativity [22] and with the center problem for vector fields [37,15].

Algebraic integrability problem has a long history. In the 19th century, the main contributors were Darboux [13], Poincaré [35,36], Painlevé [32] and Autonne [1]. They laid the foundations of a theory that has inspired a large quantity of papers, many of them published in the last twenty years. It was Darboux who gave a bound on the number of invariant integral algebraic curves of a polynomial differential equation that, when it is exceeded, implies the existence of a first integral. A close result was proved by Jouanolou [24] to guarantee that a foliation $\mathcal{F}$ as above has a rational first integral and that if one has enough reduced invariant curves; then the rational first integral can be computed. The existence of a first integral of that type is also equivalent to the fact that every invariant curve by $\mathcal{F}$ is algebraic and to the fact that there exist infinitely many invariant integral curves. These results have been adapted and extended to foliations on other varieties [23,24,5,19,12]. In [17], the authors gave an algorithm to decide about the existence of a rational first integral (and to compute it in the affirmative case) assuming that one has a well-suited set of dic($\mathcal{F}$) reduced invariant curves, where dic($\mathcal{F}$) stands for the number of dicritical divisors appearing in the resolution of $\mathcal{F}$. In the same paper, it was also shown how to get sets of invariant curves as above for foliations such that the cone of curves of the surface obtained by the resolution of the dicritical singularities is polyhedral.