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Asymptotic estimates of boundary blow-up solutions to the infinity Laplace equations ☆

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Abstract

In this paper we study the asymptotic behavior of boundary blow-up solutions to the equation $\Delta_\infty u = b(x)f(u)$ in Ω, where Δ_∞ is the ∞-Laplacian, the nonlinearity f is a positive, increasing function in $(0,\infty)$, and the weighted function $b \in C(\bar{\Omega})$ is positive in Ω and may vanish on the boundary. We first establish the exact boundary blow-up estimates with the first expansion when f is regularly varying at infinity with index $p > 3$ and the weighted function b is controlled on the boundary in some manner. Furthermore, for the case of $f(s) = s^p(1 + \tilde{c}g(s))$, with the function g normalized regularly varying with index $-q < 0$, we obtain the second expansion of solutions near the boundary. It is interesting that the second term in the asymptotic expansion of boundary blow-up solutions to the infinity Laplace equation is independent of the geometry of the domain, quite different from the boundary blow-up problems involving the classical Laplacian.

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Keywords: Infinity Laplacian; Asymptotic estimate; Boundary blow-up; First and second expansions; Comparison principle

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1. Introduction

Let Ω be a bounded C^1 domain in $\mathbb{R}^N$, $N \geqslant 2$. In this paper we study the boundary asymptotic behavior for solutions to the following problem:

$$\begin{cases} \Delta_\infty u = b(x) f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

with the weighted function b and the nonlinearity f satisfying

(H-b) $b \in C(\bar{\Omega})$, $b > 0$ in Ω;
(H-f) $f \in C[0, \infty)$, $f(0) = 0$, f is increasing in $(0, \infty)$.

Here

$$\Delta_\infty u := \sum_{i,j=1}^{N} u_{x_i} u_{x_j} u_{x_i x_j}$$

is the infinity Laplacian that arises in the work of Aronsson [1] to deduce that the infinity Laplace equation $\Delta_\infty u = 0$ is the Euler–Lagrange equation for smooth absolute minimizers. However, this elliptic operator is fully nonlinear and highly degenerate, and in general does not have smooth solutions. By adopting the viscosity solutions introduced by Crandall and Lions [13], Jensen [25] proved the equivalence of absolute minimizers and viscosity solutions of the Dirichlet problem to the infinity harmonic equation, and obtained the uniqueness of solutions. Since then, the infinity Laplace equation has been extensively studied, see e.g. [12,20,26,31,41] and the survey [2].

A natural extension for the study of the infinity Laplace equations is to consider the inhomogeneous infinity Laplace equation

$$\Delta_\infty u = h(x, u), \tag{1.2}$$

where $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. The inhomogeneous equation (1.2) with h independent of u was first studied in [32], including existence, uniqueness and stability. The game theory formulation and the PDE interpretation for the normalized infinity Laplacian were given in [39] and [33,35], respectively. When the inhomogeneous term h depends on both the variables x and u, the Dirichlet problem of (1.2) was considered in [10] for the h with the sign and the monotonicity restrictions, and in [11] without these restrictions. Some sufficient conditions on h and Ω have been proposed to ensure the existence or nonexistence of solutions. See also [34] for the comparison principle and uniqueness to a very general inhomogeneous equations containing the infinity Laplacian.

By a solution to the problem (1.1), we mean a nonnegative function $u \in C(\Omega)$ that satisfies the equation in the viscosity sense (see Section 2 for definition) and the boundary condition with $u(x) \to \infty$ as the distance function $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0$. Such a solution is called a boundary blow-up solution. It was shown in [37] that problem (1.1) admits a nonnegative solution if and only if the Keller–Osserman condition

$$\int_1^\infty \frac{dt}{\sqrt[4]{F(t)}} < \infty, \qquad F(t) = \int_0^t f(s)\, ds \tag{1.3}$$