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Compressible Navier–Stokes approximation to the Boltzmann equation

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Abstract

Even though the system of the compressible Navier–Stokes equations is not a limiting system of the Boltzmann equation when the Knudsen number tends to zero, it is the second order approximation by applying the Chapman–Enskog expansion. The purpose of this paper is to justify this approximation rigorously in mathematics. That is, if the difference between the initial data for the compressible Navier–Stokes equations and the Boltzmann equation is of the second order of the Knudsen number, so is the difference between two solutions for all time. The analysis is based on a refined energy method for a fluid-type system using the techniques for the system of viscous conservation laws.

Contents

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1. Introduction

1.1. The problem

Consider the Boltzmann equation

$$\partial_t F + \xi \cdot \nabla_x F = \frac{1}{\epsilon} Q(F, F). \tag{1.1}$$

Here $F = F(t, x, \xi) \geqslant 0$ stands for the distribution function of particles which have position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t \geqslant 0$. $\epsilon > 0$ is the Knudsen number which is proportional to the mean free path. $Q(F, H)$ in (1.1) is the Boltzmann collision operator, which for the hard sphere model takes the following form

$$\begin{aligned} Q(F, H) = \frac{1}{2} \int\limits_{\mathbb{R}^3 \times \mathbb{S}^2_+} & \big(F(\xi')H(\xi'_*) + F(\xi'_*)H(\xi') - F(\xi)H(\xi_*) \\ & - F(\xi_*)H(\xi)\big)\big|(\xi - \xi_*) \cdot \sigma\big| \, d\xi_* \, d\sigma, \end{aligned}$$

where $\mathbb{S}^2_+ = \{\sigma \in \mathbb{S}^2\colon (\xi - \xi_*) \cdot \sigma \geqslant 0\}$ and (ξ, ξ_*), and (ξ', ξ'_*), denote velocities of two particles before and after an elastic collision respectively, satisfying

$$\xi' = \xi - \big[(\xi - \xi_*) \cdot \sigma\big]\sigma, \qquad \xi'_* = \xi_* + \big[(\xi - \xi_*) \cdot \sigma\big]\sigma,$$

coming from the conservation of momentum and energy

$$\xi + \xi_* = \xi' + \xi'_*, \qquad |\xi|^2 + |\xi_*|^2 = \big|\xi'\big|^2 + \big|\xi'_*\big|^2.$$

Consequently, $|\xi - \xi_*| = |\xi' - \xi'_*|$ holds.

The Boltzmann equation (1.1) is closely related to the fluid dynamical systems, cf. [4,7–9,13–15,19,25,45,47–49] and the references cited therein for formal derivations and some mathematical analysis. To illustrate the relationship between the Boltzmann equation (1.1) and the fluid dynamical system, we adopt the argument developed in [40], which in some sense can be viewed as a unification of the classical Hilbert and Chapman–Enskog expansions. Let $F(t, x, \xi)$ be the solution to the Boltzmann equation (1.1), decompose it into the summation of the macroscopic (or fluid) part represented by the local Maxwellian $\mathbf{M} = \mathbf{M}(t, x, \xi) = \mathbf{M}_{[\rho(t,x), u(t,x), \theta(t,x)]}(\xi)$, and the microscopic (or non-fluid) part denoted by $\mathbf{G} = \mathbf{G}(t, x, \xi)$ as

$$F(t, x, \xi) = \mathbf{M}(t, x, \xi) + \mathbf{G}(t, x, \xi).$$