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WKB analysis of the Schrödinger–KdV system

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Dedicated to Professor I-Liang Chern on his 60th birthday

Abstract

We consider the behavior of solutions to the water wave interaction equations in the limit $\varepsilon \to 0^+$. To justify the semiclassical approximation, we reduce the water wave interaction equation into some hyperbolic-dispersive system by using a modified Madelung transform. The reduced system causes loss of derivatives which prevents us to apply the classical energy method to prove the existence of solution. To overcome this difficulty we introduce a modified energy method and construct the solution to the reduced system.

MSC: 35Q53; 35L60

Keywords: Zero dispersion limit; WKB analysis; System of dispersive equations; Well-posedness

1. Introduction

The purpose of this paper is to study the zero dispersion limit or WKB approximation of solutions to the water wave interaction equations

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$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{\varepsilon^2}{2}\partial_x^2 u^\varepsilon = \alpha u^\varepsilon v^\varepsilon + \beta\left|u^\varepsilon\right|^2 u^\varepsilon, & (t,x)\in\mathbb{R}^+\times\mathbb{R}, \\ \partial_t v^\varepsilon + \partial_x^3 v^\varepsilon + \partial_x (v^\varepsilon)^2 = \gamma\partial_x\left(\left|u^\varepsilon\right|^2\right), & (t,x)\in\mathbb{R}^+\times\mathbb{R}, \\ u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp\left(\dfrac{i}{\varepsilon}S_0^\varepsilon(x)\right), \quad v^\varepsilon(0,x) = v_0^\varepsilon(x), & x\in\mathbb{R}, \end{cases} \tag{1}$$

where $u^\varepsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$ and $v^\varepsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are unknown functions, $A_0^\varepsilon : \mathbb{R} \to \mathbb{C}$ and $S_0, v_0^\varepsilon : \mathbb{R} \to \mathbb{R}$ are given functions, and α, β, γ are real constants. Throughout this paper we always assume that $\beta > 0$, which corresponds to the defocusing nonlinear Schrödinger equation when the coupling coefficient α vanishes, i.e., $\alpha = 0$. The parameter ε is analogous to Planck's constant in the quantum mechanics.

It is well known that a nonlinear interaction between long and short waves can occur strongly if the phase velocity of the long wave coincides with the group velocity of the short wave. Under the assumption of long wave short wave resonance, Benny [3] proposed several systems of dispersive equations. One of the systems is given by (1) which describes an interaction phenomenon between the long and short waves arising in various physical situations such as an electron-plasma, ion field interaction and the water wave theory. In (1), the short wave is described by the Schrödinger type equation and the long wave is described by KdV type equation. The reader is referred to Kawahara–Sugimoto–Kakutani [13] for the physical background of (1) in the theory of capillary-gravity waves.

Concerning the mathematical issues for (1), the time local well-posedness for (1) has been studied by many authors where the time interval of solution depends on the parameter ε, see [2,8,17]. Recently, Wang–Cui [18] proved the local well-posedness for (1) in $L^2 \times H^{-1}$. Their proofs heavily depend on the dispersive properties of the Schrödinger equation. Therefore the time interval of solution to (1) depends on ε. In this paper we consider the semiclassical limit as $\varepsilon \to 0$ to solution to (1). To this end, we have to prove the existence of solution to (1) in some time interval independent of $\varepsilon \in (0, 1]$. Therefore our first task is to derive this existence result.

There are two approaches to justify the semiclassical approximation. Concerning the more detail for the semiclassical or WKB approximation, the reader is referred to the books [5,19] and references therein. The first approach is to use Madelung's transform defined by

$$u^\varepsilon(t,x) = \sqrt{\rho^\varepsilon(t,x)}\exp\left(\frac{i}{\varepsilon}S^\varepsilon(t,x)\right),$$

where $\rho^\varepsilon = |u^\varepsilon|^2$ and S^ε are real-valued functions. According to this design, the first equation in (1) is rewritten as

$$\begin{aligned} &\left(-2\rho^\varepsilon\partial_t S^\varepsilon - \rho^\varepsilon\left(\partial_x S^\varepsilon\right)^2 - 2\alpha\rho^\varepsilon v^\varepsilon - 2\beta\left(\rho^\varepsilon\right)^2\right) \\ &\quad + \varepsilon\left(i\partial_t\rho^\varepsilon + i\partial_x\rho^\varepsilon\partial_x S^\varepsilon + i\rho^\varepsilon\partial_x^2 S^\varepsilon\right) + \varepsilon^2\left(-\frac{1}{4}\frac{(\partial_x\rho^\varepsilon)^2}{\rho^\varepsilon} + \frac{1}{2}\partial_x^2\rho^\varepsilon\right) = 0. \end{aligned}$$

We split the above equation into the following two equations: