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Higher order shadow waves and delta shock blow up in the Chaplygin gas

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Abstract

The introductory part of this paper contains an overview of known results about elementary and delta shock solutions to Riemann problem for well known Chaplygin gas model (nowadays used in cosmological theories for dark energy) in terms of entropic shadow waves. Shadow waves are introduced in [17] and they are represented by shocks depending on a small parameter ε with unbounded amplitudes having a distributional limit involving the Dirac delta function. In a search for admissible solutions to all possible cases of mutual interactions of waves arising from double Riemann initial data we found same cases that cannot be resolved with already known types of elementary or shadow wave solutions. These cases are resolved by introducing a sequence of higher order shadow waves depending on integer powers of ε. It is shown that such waves have a distributional limit but only until some finite time T.

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1. Introduction

Many conservation law systems posses formal delta (or singular) shock wave solutions but only some of them are physically relevant. The most common admissibility criteria used to select

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a proper solution of that type is overcompressibility. A wave is called overcompressive if all characteristics from both sides run into the shock. That resembles a mass concentration that goes to infinity at some points (see [3] for a real model of such a process). In paper [17], the author introduces shadow wave solution to conservation law systems. They are represented by nets of piecewise constant function for time variable t fixed parametrized by $\varepsilon > 0$. Some of these constants are of order ε^{-1} but supported by volumes of order $\varepsilon \ll 1$ so they stay bounded in L^1_{loc}-sense with respect to x-variable for each t. They contain delta and singular shocks as special but the most important examples. Their construction enables one to easily check admissibility of the obtained solution by using the Lax (semi)convex entropy – entropy flux pair for the given system by using standard Rankine–Hugoniot conditions. In almost all cases from literature where delta or singular shocks appear, these admissibility conditions are proved to be good enough for selecting a unique (and physically relevant) delta shock (singular shock or shadow) wave solution (see [17]). Another strong point of the shadow wave theory is that the interaction problems can be easily approached. That is due to the fact that shadow waves resemble a well known and efficient Wave Front Tracking procedure (see [4] for example).

In this paper, we face the following problem. After a successful use of both (equivalent in this case also) admissibility conditions for finding a unique solution to any Riemann problem. We look for a solution to a double Riemann problem (used for investigation of wave interactions). There are cases with no admissible solution in the class of elementary and shadow waves. In fact, there exists a shadow wave weak solution but it is not admissible in the above sense. We tried to overcome that problem by introducing a sequence of shadow waves each new one parametrized by some new, smaller parameter than previous ones. Let us explain a bit that procedure: We fix ε (the initial shadow wave parameter) and solve the new initial problem obtained in such a way by using elementary waves and shadow waves parametrized now by a new $\varepsilon_1 \ll \varepsilon$. New shadow wave is said to have order one. At a new interaction time we fix ε_1 and introduce a new parameter ε_2 for shadow waves of order two and solve the initial problem. And so on. The introduction of each new parameter increases the shadow wave order by one. Later on, it will be shown that it is enough to use a sequence of parameters $\varepsilon_i = \varepsilon^{i+1}$, $i = 1, 2, \ldots$. That procedure resembles well known weak asymptotic methods (let us just mention two in a huge set of papers dealing with similar objects [13] and [7]).

The system that we investigate is the Chaplygin gas model. Some of the cosmology theories use it as a model of the so called dark energy of the Universe. It models a compressible fluid with the pressure inversely proportional to the gas energy density, $p = -A/\rho$, for some $A > 0$ (see [11] for physical explanations). There are also more recent models with the pressure defined by $p = -A/\rho^\alpha$, $0 < \alpha \leqslant 1$, with the first one (up to our knowledge) introduced in [1], that are called generalized Chaplygin gas. In these models there is a significant mathematical difference between the cases $\alpha = 1$ and $\alpha \in (0, 1)$. The first case is analyzed here, while the second one is considered in [18].

The system modeling Chaplygin gas consists of mass and momentum conservation laws

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0 \\ &\partial_t(\rho u) + \partial_x\left(\rho u^2 - \frac{A}{\rho}\right) = 0, \end{aligned}$$

where u denotes the velocity of the gas. In this paper we shall fix $A = 1$ and use the momentum variable $q = \rho u$,