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Characterization of the generic unfolding of a weak focus ☆

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ABSTRACT

In this paper we prove that the orbital class of a generic real analytic family unfolding a weak focus is determined by the conjugacy class of its Poincaré monodromy and *vice versa*. We solve the *embedding problem* by means of quasiconformal surgery on the formal normal form. The surgery yields an integrable abstract almost complex 2-manifold equipped with an elliptic foliation. The monodromy of the latter coincides with the second iterate of a germ of prescribed family of real analytic diffeomorphisms undergoing a flip bifurcation.

1. Introduction

We study a germ of real analytic planar family of ordinary differential equations depending on one *real* parameter η, with a singular point at the origin. We will assume that such family is linearly equivalent to

$$\begin{aligned}\dot{x} &= \alpha(\eta)x - \beta(\eta)y + \sum_{j+k\geqslant 2} b_{jk}(\eta)x^j y^k,\\ \dot{y} &= \beta(\eta)x + \alpha(\eta)y + \sum_{j+k\geqslant 2} c_{jk}(\eta)x^j y^k.\end{aligned} \tag{1.1}$$

The dots represent differentiation with respect to a real variable t called time. The functions $\alpha(\eta), \beta(\eta)$ are real analytic and $\alpha(0) = 0$ but $\beta(0) \neq 0$. From the general theory this family unfolds either a *center* (integral trajectories are closed curves with interiors containing the origin) or a *weak*

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focus. Weakness means that the convergence of integral curves to the origin is slower than that of logarithmic spirals of *strong foci*. Although our techniques apply to centers, we will assume in the sequel that (1.1) unfolds a weak focus (strong foci are linearizable).

We will assume also that (1.1) is *generic:* $\alpha'(0) \neq 0$. The genericity assumption and the implicit function theorem permit to take $\varepsilon := \frac{\alpha}{\beta}$ as a new parameter via a time scaling $t \mapsto \beta(\varepsilon)t$. The parameter ε is called *canonical* because it is an invariant under analytic changes (cf. Proposition 2.3). The eigenvalues of the linearization matrix of the vector field at the origin become $\varepsilon + i$ and $\varepsilon - i$ and then the singular point is *elliptic* or *monodromic* (cf. [21]). The foliation is described locally by the unfolding of the Poincaré first-return map of the positive Ox-axis, $\mathcal{P}_\varepsilon : (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ called also (Poincaré) monodromy. It is known that the germ of this map is analytic and can be extended to an analytic diffeomorphism

$$\mathcal{P}_\varepsilon : (\mathbb{R}, 0) \to (\mathbb{R}, 0). \tag{1.2}$$

With an elliptic point of (1.1) the *displacement function* $\mathcal{P}_\varepsilon(x) - x$ may be associated. Isolated roots of the displacement function correspond to limit cycles of the vector field. From the general theory, the normal form of the displacement function always starts with an odd-power term x^{2k+1}, $k \geqslant 1$. The coefficient $\ell_k(\varepsilon)$ of this term is called the k-th Lyapunov constant. Eq. (1.1) is of *order k* provided $\ell_j(0) = 0$ for all $j = 1, \dots, k-1$ but $\ell_k(0) \neq 0$. In this case the system undergoes a generic *Hopf bifurcation of codimension k*. It is obvious that $k-1$ additional real parameters are required to describe this bifurcation. (Evidently, the ℓ_k's depend on these parameters as well.) In this situation, exactly k limit cycles appear and merge with the origin as the parameters tend to the *bifurcation value* (typically 0).

The definition of the order is independent of the choice of the transversal and the coordinate system: the order is a geometric invariant of the germ of the analytic family. By definition, the order of an integrable field is equal to $+\infty$ (cf. [21]). For instance, a germ of planar analytic monodromic family depending on parameters unfolds a center at zero if and only if each Lyapunov constant vanishes at the bifurcation value. That is why centers are also called *weak foci of infinite order.*

In this manuscript we will assume that (1.1) is of order 1. The Lyapunov first constant can be then explicitly computed in terms of the coefficients of the field:

$$\ell_1 = 3b_{30} + b_{12} + c_{21} + 3c_{03} + \frac{1}{\beta}\big[b_{11}(b_{20} + b_{02}) - c_{11}(c_{20} + c_{02}) - 2b_{20}c_{20} + 2b_{02}c_{02}\big].$$

In this case the system exhibits a generic Hopf bifurcation (generic coalescence of a focus with a limit cycle). The Hopf bifurcation is *subcritical* if the cycle is present on negative values of ε. It is *supercritical* otherwise. Whether a Hopf bifurcation is subcritical or supercritical can be found from the sign of the first Lyapunov coefficient. Positive sign of $\ell_1(0)$ indicates a subcritical Hopf bifurcation and negative sign of $\ell_1(0)$ corresponds to a supercritical Hopf bifurcation, cf. Proposition 2.3.

A question that arises naturally is whether the germ of the monodromy defines the analytic equivalence class of the real foliation. The natural way to answer this question is via complexification of time, coordinates and parameter. Without loss of generality, the domain of the parameter will be assumed to be a standard open complex disk $\mathbb{D}_\varepsilon$ of small diameter. After complexification of coordinates x, y, we define the variables $z = x + iy$, $w = x - iy$ and express the complexified family of vector fields in these new coordinates. The complexified system takes the form

$$F_\varepsilon = \mathbf{U}_\varepsilon \frac{\partial}{\partial z} + \mathbf{V}_\varepsilon \frac{\partial}{\partial w} \tag{1.3}$$

such that $\mathbf{U}_\varepsilon(z, w) = \overline{\mathbf{V}_{\bar\varepsilon}(\overline{w}, \bar z)}$, where $z \mapsto \bar z$ is the standard complex conjugation. By the Hadamard–Perron Theorem for holomorphic flows (cf. [8], p. 106), the field (1.3) has two holomorphic separatrices tangent to the axis of coordinates.

By an *antiholomorphic involution* or *real structure* of a complex manifold M we mean an antiholomorphic map $\sigma : M \to M$ such that $\sigma \circ \sigma = id_M$. Not every pair (M, σ) can be obtained by