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Boundary layer analysis of the Navier–Stokes equations with generalized Navier boundary conditions

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ABSTRACT

We study the weak boundary layer phenomenon of the Navier–Stokes equations with generalized Navier friction boundary conditions, $u \cdot \boldsymbol{n} = 0$, $[\mathbf{S}(u)\boldsymbol{n}]_{\tan} + \mathcal{A}u = 0$, in a bounded domain in $\mathbb{R}^3$ when the viscosity, $\varepsilon > 0$, is small. Here, $\mathbf{S}(u)$ is the symmetric gradient of the velocity, u, and $\mathcal{A}$ is a type $(1,1)$ tensor on the boundary. When $\mathcal{A} = \alpha I$ we obtain Navier boundary conditions, and when $\mathcal{A}$ is the shape operator we obtain the conditions, $u \cdot \boldsymbol{n} = (\operatorname{curl} u) \times \boldsymbol{n} = 0$. By constructing an explicit corrector, we prove the convergence, as ε tends to zero, of the Navier–Stokes solutions to the Euler solution both in the natural energy norm and uniformly in time and space.

1. Introduction

The flow of an incompressible, constant-density, constant-viscosity Newtonian fluid is described by the Navier–Stokes equations,

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + \big(u^\varepsilon \cdot \nabla\big) u^\varepsilon + \nabla p^\varepsilon = f & \text{in } \Omega \times (0,T), \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \Omega \times (0,T), \\ u^\varepsilon\big|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

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The fluid is contained in the bounded domain, $\Omega \subset \mathbb{R}^3$, with smooth boundary, Γ. The parameter, $\varepsilon > 0$, is the viscosity and $T > 0$ is fixed (see Theorem 2.2). The equations are to be solved for the velocity of the fluid, u^ε, and pressure, p^ε, given the forcing function, f, and initial velocity, u_0. The regularity of Γ, f, and u_0 that we assume is specified in (1.7), but our emphasis is not on optimal regularity requirements. We also impose Navier boundary conditions on u^ε, described below, which include the impermeable condition, $u^\varepsilon \cdot \boldsymbol{n} = 0$.

When $\varepsilon = 0$, we formally obtain the Euler equations,

$$\begin{cases} \dfrac{\partial u^0}{\partial t} + (u^0 \cdot \nabla) u^0 + \nabla p^0 = f & \text{in } \Omega \times (0, T), \\ \operatorname{div} u^0 = 0 & \text{in } \Omega \times (0, T), \\ u^0 \cdot \boldsymbol{n} = 0 & \text{on } \Gamma \times (0, T), \\ u^0|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\boldsymbol{n}$ is the outer unit normal vector on Γ.

The Euler equations, being first-order, need only the impermeable boundary condition, $u^0 \cdot \boldsymbol{n} = 0$, reflecting no entry or exit of fluid from the domain. No-slip boundary conditions, $u^\varepsilon = 0$ on Γ, are those most often prescribed for the Navier–Stokes equations. This, of necessity, leads to a discrepancy between u^ε and u^0 at the boundary, resulting in boundary layer effects. Prandtl [46] was the first to make real progress on analyzing these effects, and much of a pragmatic nature has been discovered, but to this day the mathematical understanding is woefully inadequate. (See [15,9,40,17] for reviews of the mathematical literature. See [24], which builds on linear results of [18,19,25], for ill-posedness of Prandtl's boundary layer equations; [19] gives a review of earlier ill-posedness results. See [31,57, 60,10,33,34,49] for conditional results on convergence in the vanishing viscosity limit.)

In part because of these difficulties with no-slip boundary conditions, and in part because of very real physical applications, researchers have turned to other boundary conditions. Of particular interest are boundary conditions variously called Navier friction, Navier slip, or simply Navier boundary conditions (other names have been used as well). These boundary conditions can be written as

$$u^\varepsilon \cdot \boldsymbol{n} = 0, \qquad \left[\mathbf{S}(u^\varepsilon)\boldsymbol{n} + \alpha u^\varepsilon\right]_{\tan} = 0 \quad \text{on } \Gamma, \tag{1.3}$$

where

$$\mathbf{S}(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^{\mathsf{T}}\right) = \left(\frac{1}{2}\frac{\partial u_j}{\partial x_i} + \frac{1}{2}\frac{\partial u_i}{\partial x_j}\right)_{1 \leqslant i,j \leqslant 3}, \quad \text{for } u = (u_1, u_2, u_3). \tag{1.4}$$

Here (x_1, x_2, x_3) (or (x, y, z) in Section 3), denotes the Cartesian coordinates of a point $\boldsymbol{x} \in \mathbb{R}^3$, α is the (positive or negative) friction coefficient, which is independent of ε. The notation $[\cdot]_{\tan}$ in (1.3) denotes the tangential components of a vector on Γ.

In this paper, we use the generalization of (1.3),

$$\begin{cases} u^\varepsilon \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \left[\mathbf{S}(u^\varepsilon)\boldsymbol{n}\right]_{\tan} + \mathcal{A} u^\varepsilon = 0 & \text{on } \Gamma, \end{cases} \tag{1.5}$$

of the Navier boundary conditions. Here, $\mathcal{A}$ is a type $(1,1)$ tensor on the boundary having at least C^2-regularity. In coordinates on the boundary, $\mathcal{A}$ can be written in matrix form as $\mathcal{A} = (\alpha_{ij})_{1 \leqslant ij \leqslant 2}$. Note that u^ε lies in the tangent plane, as does $\mathcal{A} u^\varepsilon$.

It is easy to see that when $\mathcal{A} = \alpha I$, the product of a function α on Γ and the identity tensor, the generalized Navier boundary conditions, (1.5), reduce to the usual Navier friction boundary conditions, (1.3). In fact, the analysis using a general $\mathcal{A}$ in place of αI is changed only slightly from using αI with α a constant (we say a bit more on this in Remark 2.4).