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Hölder continuous solutions for fractional differential equations and maximal regularity

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We characterize the existence and uniqueness of solutions of an abstract fractional differential equation with infinite delay in Hölder spaces.

1. Introduction

In this paper, we consider the following fractional differential equation with infinite delay

$$D^\beta u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where A is a closed linear operator defined on a Banach space X, $a \in L^1(\mathbb{R}_+)$ is a scalar-valued kernel, $f \in C^\alpha(\mathbb{R}; X)$, $0 < \alpha < 1$, and the fractional derivative for $\beta > 0$ is taken in the sense of Caputo.

Fractional differential equations have been used by many researchers to adequately describe the evolution of a variety of physical and biological processes. Examples include studies in

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electrochemistry, electromagnetism, viscoelasticity, rheology, among other. See, for instance [1,26,31,34] for further details.

When $\beta = 1$ in Eq. (1.1), we obtain the equation with infinite delay

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t), \quad t \in \mathbb{R}. \tag{1.2}$$

Equations of this kind arise, for example, in the study of heat flow in materials with memory as well as in some equations of population dynamics or in viscoelasticity. In such applications the operator A is typically the Laplacian, the elasticity operator, the Stokes operator, or the biharmonic Δ^2, among other. See [38] and [40, Chapter III, Section 13] for further details.

In [4], Arendt, Batty and Bu study the existence and uniqueness of Hölder continuous solutions to Eq. (1.1) in the case $\beta = 1$ and $a \equiv 0$, that is,

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R}. \tag{1.3}$$

These authors introduce the notion of C^α-*multipliers* and prove an operator-valued Fourier multiplier theorem for Hölder spaces on the line. As a consequence, the authors obtain a characterization, by means of a simple resolvent estimate of the underlying operator, of the *well-posedness* of (1.3), in the sense that there exists a unique classical solution to Eq. (1.3) in Hölder spaces.

Using the results of Arendt, Batty and Bu [4] about vector-valued Fourier multipliers, some characterizations of the existence and uniqueness of solutions (on the real line) for several classes of equations in Hölder spaces, have been obtained in the last years. See [12,18,30,39] for further information.

Existence of Hölder continuous solutions to fractional differential equations in the form of (1.1) has been studied for example, by Clement, Gripenberg and Londen using the method of the sum of Da Prato and Grisvard [16]. See moreover El-Sayed and Herzallah [21–23] and references therein. Other approaches to the existence and uniqueness of solutions to fractional differential equations can be found, for example, in [6,9,13,17,20,32]. The obtained results give sufficient conditions to the existence and uniqueness of Hölder continuous solutions to equations in the form of (1.1), but leave as an open problem to *characterize* the existence and uniqueness of Hölder solutions to fractional differential equations.

Characterizations of the existence and uniqueness of solutions to the linear problem (1.1) have been studied only on periodic vector-valued Lebesgue spaces, $L^p_{2\pi}(\mathbb{R}; X)$, $1 < p < \infty$ (where X is a *UMD* space) and in the scale of periodic Besov spaces $B^s_{p,q}([0, 2\pi]; X)$ (and therefore on periodic Hölder space $C^s([0, 2\pi]; X)$) by Bu [10,11]. The main tool in these results are two operator-valued Fourier multipliers theorems of Arendt and Bu [5,7] on periodic vector-valued spaces $L^p_{2\pi}(\mathbb{R}; X)$, $1 < p < \infty$, and $B^s_{p,q}([0, 2\pi]; X)$. Using these results on operator-valued multipliers, other fractional differential equations on periodic vector-valued spaces have been recently studied in [29,33]. In all these results is obtained a relation between the existence and uniqueness of solutions to fractional differential equations and the $\mathcal{R}$-boundedness of a sequence of operators (see [19,27,41]).

In this paper, we apply the method of operator-valued Fourier multipliers on the line (see [4]) to characterize the well-posedness (or maximal regularity) of the fractional differential equation (1.1) in $C^\alpha(\mathbb{R}; X)$, the vector-valued Hölder spaces for $0 < \alpha < 1$. More specifically, we show in Theorem 3.7 that if a is a 2-regular kernel, then the problem (1.1) is C^α-well posed if and only if

$$\frac{(i\eta)^\beta}{1+\tilde{a}(\eta)} \in \rho(A), \quad \text{for all } \eta \in \mathbb{R} \quad \text{and} \quad \sup_{\eta\in\mathbb{R}} \left\| \frac{(i\eta)^\beta}{1+\tilde{a}(\eta)} \left(\frac{(i\eta)^\beta}{1+\tilde{a}(\eta)} - A \right)^{-1} \right\| < \infty,$$

where $\tilde{a}$ denotes the Fourier transform of a (more precisely of their extension to $\mathbb{R}$ by setting them equal to 0 on $(-\infty, 0)$).