J. Differential Equations 255 (2013) 3412–3433

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

# Existence and non-existence of solutions for p-Laplacian equations with decaying cylindrical potentials

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ARTICLE INFO

*Article history:*
Received 31 October 2011
Available online 12 August 2013

*Keywords:*
p-Laplacian operator
Critical potential
Cylindric symmetric solution

ABSTRACT

In this paper we deal with the problem

$$-\Delta_p u + \frac{|u|^{p_*(s)-2}u}{|y|^s} = |u|^{m-2}u, \quad u \in D^{1,p}(\mathbb{R}^N;\mathbb{R})$$

where $1 < p < N$, $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $\Delta_p$ is p-Laplacian operator, $p_*(s) = \frac{p(N-s)}{N-p}$ and $p^* = \frac{pN}{N-p}$. Combining a version of the concentration compactness result by Solimini, Hardy–Sobolev type inequality with the Mountain Pass Theorem, existence of non-trivial solutions is obtained. Decay properties of these solutions are showed by applying Vassilev results. Pohozaev type identities are established in order to get non-existence results.



## 1. Introduction

Existence, non-existence and qualitative properties of non-trivial solutions for nonlinear elliptic equations with singular potentials were recently studied by several authors, most of them were motivated by the search for solitary waves of nonlinear equations of the Schrödinger or Klein–Gordon type and for extremals to Hardy–Sobolev inequalities, see [1,4–7,9,21,24,26–32] and references therein.

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1 Supported in part by CPNq/Brazil and Capes.
2 Supported in part by CPNq/Brazil and INCTMat-CNPq/Brazil.

0022-0396/$ – see front matter 
http://dx.doi.org/10.1016/j.jde.2013.07.037

In this paper we are concerned with the existence, non-existence and regularity of non-trivial solutions for a class of nonlinear elliptic differential equations in $\mathbb{R}^N$ of type

$$-\Delta_p u + \frac{|u|^{p_*(s)-2}u}{|y|^s} = |u|^{m-2}u \quad \text{in } \left(\mathbb{R}^N \setminus \{0\}\right) \times \mathbb{R}^{N-k}, \tag{1}$$

where $1 < p < N$, $2 \leqslant k < N$, $0 < s \leqslant N - \frac{N-p}{p}$, $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $\Delta_p$ is p-Laplacian operator, $p_*(s) = \frac{p(N-s)}{N-p}$ and $p^* = \frac{pN}{N-p}$.

Badiale and Tarantello in [9] obtained the following Hardy–Sobolev inequality

$$\int_{\mathbb{R}^N} \frac{|u|^{p_*(s)}}{|y|^s}\,dx \leqslant C\left(\int_{\mathbb{R}^N} |\nabla u|^p\right)^{\frac{p_*(s)}{p}} \tag{2}$$

for all $u \in D^{1,p}(\mathbb{R}^N)$, where $C$ is a positive constant independent of $u$. This inequality generalizes the analogous inequality obtained by Caffarelli, Kohn and Nirenberg in [17] for the case $k = N$, by an interpolation between the Hardy inequality, where $s = p$, and the Sobolev inequality corresponding to $s = 0$. As a consequence of this result they studied the problem

$$-\Delta u = V\left(|y|\right)|u|^{p-2}u, \quad u > 0 \text{ on } \mathbb{R}^N, \tag{3}$$

which was proposed by two astrophysicists, Bertin and Ciotti, as a model describing the dynamics of galaxies. Applying the Concentration Compactness Principle due to Solimini [25], Badiale, Benci and Rolando in [5] established an existence result considering more general nonlinearities. Actually, they applied their results to get solitary waves of nonlinear equations of the Schrödinger or Klein–Gordon type. In [6], Badiale, Guida and Rolando extended this result treating the case $V(|y|) = |y|^{-\alpha}$. Badiale and Rolando in [8] dealt the case when the potential $V$ is subhomogeneous and exhibiting some symmetries of the type $V(x) = V(|y_1|, \ldots, |y_k|)$, where $x = (y_1, \ldots, y_k, z) \in \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k} \times \mathbb{R}^{N_0}$. More references involving the Laplacian operator with cylindrical symmetry can be seen, for instance, in [10,12,18,23] and references therein.

In 2003, Xuan in [32] obtained the existence of multiple weak solutions results for the following subcritical Dirichlet problem involving p-Laplacian operator in a smooth bounded domain containing 0 in its interior

$$\begin{cases} -\Delta_p u = \lambda|u|^{r-2}u + \dfrac{|u|^{q-2}u}{|y|^s} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $1 < r < p < q \leqslant p_*(s)$ and $\lambda > 0$. Recently, Bhakta and Biswas in [11] proved existence, non-existence and regularity results to the above problem with $\lambda = 0$, and $\Omega$ an open smooth domain. Assunção, Carrião and Miyagaki in [3] and Bouchekif and Mokhtar in [14] studied a class of subcritical degenerate quasilinear problem involving singularities on a half-space of $\mathbb{R}^N$ and $\mathbb{R}^N$, respectively.

In this paper, in addition to extend previous results for a class of quasilinear problems involving p-Laplacian operator, we deal with more general nonlinearities with critical growth. The main difficulty is that besides the lack of the compactness, the left-hand side term of Eq. (1) is inhomogeneous in $u$, and also to prove some weak convergence we need to establish the convergence of the gradient of $u$, almost everywhere.

**Notation.**

- We denote by $p^* := \frac{pN}{N-p}$ the critical exponent for the Sobolev embedding in dimension $N \geqslant p$. And by $p_* := p_*(s) = \frac{p(N-s)}{N-p}$ the critical Hardy–Sobolev exponent.