J. Differential Equations 253 (2012) 379–399

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

# Nonautonomous saddle-node bifurcations: Random and deterministic forcing

V. Anagnostopoulou, T. Jäger *

*Department of Mathematics, TU Dresden, Germany*

ARTICLE INFO

*Article history:*
Received 26 May 2011
Revised 16 January 2012
Available online 17 April 2012

*MSC:*
primary 34C23
secondary 34F10, 37H20

ABSTRACT

We study the effect of external forcing on the saddle-node bifurcation pattern of interval maps. By replacing fixed points of unperturbed maps by invariant graphs, we obtain direct analogues to the classical result both for random forcing by measure-preserving dynamical systems and for deterministic forcing by homeomorphisms of compact metric spaces. Additional assumptions like ergodicity or minimality of the forcing process then yield further information about the dynamics.

The main difference to the unforced situation is that at the critical bifurcation parameter, two alternatives exist. In addition to the possibility of a unique neutral invariant graph, corresponding to a neutral fixed point, a pair of so-called pinched invariant graphs may occur. In quasiperiodically forced systems, these are often referred to as 'strange non-chaotic attractors'. The results on deterministic forcing can be considered as an extension of the work of Novo, Núñez, Obaya and Sanz on nonautonomous convex scalar differential equations. As a by-product, we also give a generalisation of a result by Sturman and Stark on the structure of minimal sets in forced systems.



## 1. Introduction

An important question which arises frequently in applications is that of the influence of external forcing on the bifurcation patterns of deterministic dynamical systems. This has been one of the main motivations for the development of random dynamical systems theory (compare [1, Chapter 9]), and the description of the nonautonomous counterparts of the classical bifurcation patterns is one

* Corresponding author.
*E-mail addresses:* vasso.anagnostopoulou@gmail.com (V. Anagnostopoulou), Tobias.Oertel-Jaeger@tu-dresden.de (T. Jäger).

0022-0396/$ – see front matter 
http://dx.doi.org/10.1016/j.jde.2012.03.016

of the principal goals of nonautonomous bifurcation theory. The different types of forcing processes which are of interest range from deterministic systems like quasiperiodic motion or, more generally, strictly ergodic dynamics on the one side to random or stochastic processes like Brownian motion (white noise) at the other end of the spectrum. The reader is referred to [1, Section 9] for a good introduction to the topic and to [2–7] for more recent developments and further references.

Our aim here is to consider one of the simplest types of bifurcations, namely saddle-node bifurcations of interval maps or scalar differential equations. Given a forcing transformation $\omega : \Theta \to \Theta$, where $\Theta$ is either a measure space or a topological space, we study skew product maps of the form

$$f(\theta, x) : \Theta \times [a, b] \to \Theta \times [a, b], \quad (\theta, x) \mapsto \big(\omega(\theta), f_\theta(x)\big), \tag{1.1}$$

where $\omega : \Theta \to \Theta$ is called the *forcing process* or *base transformation*. The bifurcating objects we concentrate on are *invariant graphs*, that is, measurable functions $\varphi : \Theta \to [a, b]$ which satisfy

$$f_\theta\big(\varphi(\theta)\big) = \varphi\big(\omega(\theta)\big) \tag{1.2}$$

for all (or at least almost all) $\theta \in \Theta$. Suppose we are given a parameter family $(f_\beta)_{\beta\in[0,1]}$ of maps of the form (1.1) and a region $\Gamma \subseteq \Theta \times [a, b]$. Then our objective is to provide a criterion for the occurrence of saddle-node bifurcations (of invariant graphs) inside of $\Gamma$. More precisely, we show the existence of a critical bifurcation parameter $\beta_c$ such that:

- If $\beta < \beta_c$, then $f_\beta$ has two invariant graphs in $\Gamma$.
- If $\beta > \beta_c$, then $f_\beta$ has no invariant graphs in $\Gamma$.
- If $\beta = \beta_c$, then $f_\beta$ has either one or two invariant graphs in $\Gamma$. If there exist two invariant graphs, then these are 'interwoven' in a certain sense (*pinched*, Section 3).

Apart from some mild technical conditions, the crucial assumptions we need to establish statements of this type are the monotonicity of the fibre maps $f_\theta$, both with respect to $x$ and to the parameter $\beta$, and their convexity inside of the considered region $\Gamma$ (see Theorems 4.1 and 6.1).

Nonautonomous saddle-node bifurcations of this type have been studied previously in [3,4] for nonautonomous scalar convex differential equations over a strictly ergodic base flow and in [8,9] for quasiperiodically forced interval maps. In all cases, the proofs hinge on a convexity argument used to control the number of invariant graphs or, more or less equivalently, minimal sets in the system. This simple, but elegant and powerful idea can be traced back to Keller [10] and has later been used independently by Alonso and Obaya [11] in order to classify nonautonomous scalar convex differential equations according to the structure of their minimal sets. However, so far no systematic use of these arguments has been made in order to determine the greatest generality to which the description of nonautonomous saddle-node bifurcations can be pushed. This is the goal of the present paper. Quite surprisingly, it turns out that hardly any assumptions on the underlying forcing process are needed in order to give a fairly good description of the bifurcation pattern. We only require that the forcing transformation is invertible and that it is either a measure-preserving transformation of a probability space or a homeomorphism of a compact metric space. In the former case, we work in a purely measure-theoretic setting, such that no topological structure on the base space is required. Additional properties like ergodicity, respectively minimality, can be used in order to obtain further information about the dynamics.

As a by-product of our studies in the topological setting, we also obtain a generalisation of a result by Sturman and Stark [12] concerning the structure of invariant sets. If a compact invariant set of a minimally driven $\mathcal{C}^1$-map on a Riemannian manifold only admits negative upper Lyapunov exponents (with respect to any invariant measure supported on $M$), then $M$ is just a finite union of continuous curves (see Theorem 5.3).

The paper is organised as follows. In Section 2, we collect a number of preliminaries on forced interval maps, including the convexity result due to Keller. In Section 3, we introduce and discuss