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A singular reaction–diffusion system modelling prey–predator interactions: Invasion and co-extinction waves

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ABSTRACT

We consider a singular reaction–diffusion system arising in modelling prey–predator interactions in a fragile environment. Since the underlying ODEs system exhibits a complex dynamics including possible finite time quenching, one first provides a suitable notion of global travelling wave weak solution. Then our study focusses on the existence of travelling waves solutions for predator invasion in such environments. We devise a regularized problem to prove the existence of travelling wave solutions for predator invasion followed by a possible co-extinction tail for both species. Under suitable assumptions on the diffusion coefficients and on species growth rates we show that travelling wave solutions are actually positive on a half line and identically zero elsewhere, such a property arising for every admissible wave speeds.

1. Introduction

In this work we consider a singular diffusive prey–predator system

$$\begin{aligned} &\frac{\partial B(t,x)}{\partial t} - d\Delta B(t,x) = B(t,x)g\big(B(t,x)\big) - C(t,x), \\ &\frac{\partial C(t,x)}{\partial t} - \Delta C(t,x) = rC(t,x)\left(1 - \frac{C(t,x)}{B(t,x)}\right), \end{aligned} \tag{1}$$

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posed for $t > 0$, $x \in \mathbb{R}^N$. The underlying spatially homogeneous problem was derived in Courchamp and Sugihara [6] to model prey–predator interactions in fragile (insular) environments. The spatially structured System (1) supplemented with initial data and no-flux boundary conditions was introduced in Gaucel and Langlais [9]. Herein $B(t,x)$ (resp. $C(t,x)$) denotes the density of prey (resp. predator) at time t located at $x \in \mathbb{R}^N$. Parameter $d > 0$ represents the normalized diffusion coefficient, namely the ratio between the actual diffusivities of prey and predators. Function g stands for the intrinsic growth rate of prey while $r > 0$ is the growth rate of predators. In the original model [6] $Bg(B)$ is a logistic growth function.

Note that System (1) can also be viewed as a special case of the so-called Holling–Tanner prey–predator system, see [11] or [12],

$$\begin{aligned} \frac{\partial B(t,x)}{\partial t} - d\Delta B(t,x) &= B(t,x)g\big(B(t,x)\big) - \frac{B(t,x)C(t,x)}{\gamma + B(t,x)}, \\ \frac{\partial C(t,x)}{\partial t} - \Delta C(t,x) &= rC(t,x)\left(1 - \frac{C(t,x)}{B(t,x)}\right). \end{aligned} \tag{2}$$

From a formal point of view, System (1) is a specific cases of System (2) with $\gamma = 0$. However, as it is proved in [9], System (1) may exhibit finite time quenching (that is not the case for System (2)) so that the above mentioned formal limit turns out to be a singular like limit.

The aim of this work is to look for the existence of travelling wave solutions of predator invasion for System (1). Numerical simulations in Burie, Ducrot and Langlais (work in progress) suggested that travelling wave solutions correspond to the typical numerical response of the system to introducing a spatially localized perturbation of predators within a homogeneous population of prey resting at its carrying capacity.

Due to the finite time quenching property of the evolution system under consideration, namely System (1), one may expect that under some suitable circumstances, after the predator invasion wave both populations may vanish, that is, the solution decreases to $B = C = 0$ at a finite spatio-temporal location. The aim of this work is to understand such a qualitative property of the predator invasion waves, that is the existence of so-called sharp travelling waves. We refer to [2] and [3] for first results on the existence of sharp travelling waves in the context of degenerate reaction–diffusion equations. One also refers to [13], [14], [16] or [17] for other results and discussions on this topic in the context of degenerate Fisher–KPP equations.

Such singular travelling waves problems also arise for reaction–diffusion (and convection) equations exhibiting a finite time blow-up. These waves are sometimes called semi-finite waves. See for instance the monograph of [10] and references therein.

Let us also mention that, according to our knowledge, only little work has been done for sharp waves or semi-finite waves for reaction–diffusion systems without comparison principle. We refer to [15] for results on the existence of travelling waves for degenerate reaction–diffusion systems without comparison principle modelling bacterial pattern formation.

Note that in this work System (1) does not exhibit any diffusion degeneracy but a singular reaction term. The method we shall develop is a regularization procedure. To be more precise, one shall first look at travelling wave solutions for the Holling–Tanner model (2) and then pass to the limit as $\gamma \to 0$ to get suitable (weak) travelling wave solutions to (1). Qualitative properties of these weak solutions are carefully studied to find sufficient conditions on the parameter set ensuring the existence of either everywhere positive waves or of sharp waves that vanish at some finite spatial location.

Our work is organized as follows. Section 2 is devoted to listing our main assumptions and to stating our main results. In Section 3 the existence of travelling wave solution for the regularized Holling–Tanner system is analyzed. Section 4 is concerned with the proof of some non-existence results. Section 5 deals with passing to the limit in the regularization to provide the existence of mild solutions. Finally, Sections 6–8 are concerned with the proofs of qualitative properties of travelling wave solutions.