J. Differential Equations 254 (2013) 3329–3351

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Long range scattering for higher order Schrödinger operators

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A R T I C L E   I N F O

Article history:
Received 26 September 2012
Revised 6 January 2013
Available online 12 February 2013

MSC:
35P25
35S05
47A40
81U20

Keywords:
Higher order Schrödinger operator
Wave operator
Scattering operator

A B S T R A C T

This paper is concerned with the scattering problem for the higher order Schrödinger operator $(-\Delta)^m + V(x)$, where $V(x)$ is a long-range potential. At this moment the classical wave operators do not exist, so we provide one kind of modified wave operators and show the existence and the asymptotic completeness of such wave operators, as well as the expression of the corresponding scattering operator.

1. Introduction

We consider the higher order Schrödinger operators $H = (-\Delta)^m + V(x)$ and $H_0 = (-\Delta)^m$ (m is a positive integral number) in the space $L^2(\mathbf{R}^d)$, where Δ is Laplace operator and $V(x)$ is a long-range potential satisfying

$$\left|\partial^\alpha V(x)\right| \leqslant C_\alpha \left(1 + |x|\right)^{-\rho-|\alpha|}, \quad 1/2 < \rho < 1. \tag{1.1}$$

It is well known that the scattering problem of Schrödinger operators has received much attention [1,3,4,8,11,16]. Usually, one considers two Hamilton systems, i.e., the disturbance Hamilton system H and the free Hamilton system H_0. The potential $V(x)$ is often divided into the short-range case

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.01.022

[6, Chapter 14] and long-range case [7, Chapter 30] by decay index $\rho > 1$ and $\rho < 1$ (as in (1.1)), respectively (cf. [15]).

In the case of long-range potentials [5,18,19], the classical wave operators do not exist. For second order Schrödinger operators, Yafaev (see [18]) modified wave operators as

$$\Omega_\pm(H, H_0; J_\pm) = s - \lim_{t\to\mp\infty} e^{iHt} J_\pm e^{-iH_0 t}, \tag{1.2}$$

and proved the existence of the modified wave operators by using the smooth perturbation theory. In [7], the wave operators also were modified as $e^{itH}e^{-iW(D,t)}$, where $W(\xi,t)$ is a suitable Hamilton flow.

On the other hand, for the higher order Schrödinger operators (e.g. [2,10,21,22]), the self-adjoint and the local smoothing properties were discussed. In general, many but not all problems for the second and higher order differential operators admit similar solutions. It is difficult to indicate at what step the passage from $m = 1$ to an arbitrary m might create new difficulties. However a definite answer requires a specific analysis.

In this paper our main goal is to prove the existence and the asymptotic completeness of the modified wave operators as in (1.2) for the higher order Schrödinger operators with the long-range potential (1.1) by using the theory of smooth perturbations (see Kato [9]). The paper is organized as follows.

The purpose of scattering theory is to examine how the spectral decomposition of H_0 is changed if H_0 is perturbed to an operator H which is fairly close to H_0 at infinity. In Section 2, we first check Mourre's condition [13] for the higher order Schrödinger operators, and then show that H has no singular continuous spectrum in some bounded open interval by using Mourre's method [13] (see also [14]).

Section 3 is provided to the H-smoothness. From Section 2 we know that for $p > 1/2$, the operator $X^{-p}\chi_\Lambda(H)$ is H-smooth ($X = (1+|x|^2)^{1/2}$, χ_Λ is the characteristic function on Λ). Further, contrast with the local smoothness (cf. [2,10,12]), the operator $X^{-p}(H_0+I)^{1/2}\chi_\Lambda(H)$ ($p > 1/2$) is H-smooth (the factor $(H_0+I)^{1/2}$ is compensated here by $\chi_\Lambda(H)$). In the short-range case the limiting absorption principle is sufficient (see Section 2.1 in [19] for $m = 1$) for construction of scattering theory by the above H-smoothness ($p > 1/2$) but, for long-range potentials, one needs an additional analytical information pertaining in some sense to the critical case $p = 1/2$. To this end, we construct the following operator

$$G_{mj}u = X^{-1/2}\big(\partial_j(A_{m,0}u) - |x|^{-2}\big(\nabla(A_{m,0}u), x\big)x_j\big),$$

where $A_{m,0}$ is a suitable $m-1$ order differential operator. The key point is to employ the commuter $\mathcal{N}$ (see (3.3) below) to show the H-smoothness of G_{mj}.

Section 4 is concerned with the construction of the modified wave operators as in (1.2), in which $J_\pm$ are constructed as Fourier integral operators with the phase functions satisfying the eikonal equation.

In Section 5, we prove the main theorem, i.e., the existence of the modified wave operators given by (1.2). To this end, we use the theory of smooth perturbations to discuss the effective perturbations $T_\pm = HJ_\pm - J_\pm H_0$, which are the pseudo-differential operators with the symbol vanishing as $|x|^{-1}$ at infinity. The operators $T_\pm$ have the form of

$$T_\pm = \sum_j G_{mj}^* A_\pm^{(s)} G_{mj} + X^{-p} A_\pm^{(r)} X^{-p},$$

where $A_\pm^{(s)}$ and $A_\pm^{(r)}$ are bounded operators. In this way, we prove the existence of the modified wave operators by the H-smoothness of G_{mj} and X^{-p} ($p > 1/2$). Moreover, the asymptotic completeness of the modified wave operators and the expression of the scattering operator are also given by the standard method.