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# On the axiomatic approach to Harnack's inequality in doubling quasi-metric spaces ☆

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ABSTRACT

We introduce a novel approach towards Harnack's inequality in the context of spaces of homogeneous type. This approach, based on the so-called critical density property and doubling properties for weights, avoids the explicit use of covering lemmas and BMO.



## 1. Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\mathcal{B}_\Omega$ denote the collection of Euclidean balls $B$ such that $4B \subset \Omega$. By means of his celebrated iterative procedure J. Moser proved in [34] that positive subsolutions to homogeneous divergence-form uniformly elliptic PDEs satisfy weak reverse-Hölder inequalities of the form

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$$\sup_B u \leqslant C\left(\frac{1}{|B|}\int_{2B} u^q\,dx\right)^{\frac{1}{q}}, \quad B \in \mathcal{B}_\Omega,\ q > 0, \tag{1.1}$$

where $C$ depends on $n$, $q$, and the ellipticity constants and $|B|$ stands for Lebesgue measure of $B$. Now, if $u$ is a positive solution, the previous result applied to $1/u$ yields, in addition, the inequalities

$$\left(\frac{1}{|B|}\int_{2B} u^{-q}\,dx\right)^{-\frac{1}{q}} \leqslant C\inf_B u, \quad B \in \mathcal{B}_\Omega,\ q > 0. \tag{1.2}$$

Then, Moser's Harnack inequality for $u$, namely,

$$\sup_B u \leqslant C\inf_B u, \quad B \in \mathcal{B}_\Omega, \tag{1.3}$$

follows after proving that $\log u \in BMO(\Omega)$, whenever $u$ is a positive supersolution. Indeed, the John–Nirenberg inequality renders the equivalence between $\log u \in BMO(\Omega)$ and the existence of $q_0 > 0$ such that $u^{q_0} \in A_2(\mathcal{B}_\Omega)$ (the $A_2$ Muckenhoupt class associated to $\mathcal{B}_\Omega$, see (2.12)). Finally, the choice $q := q_0$ in (1.1) and (1.2) yields (1.3). Moser's approach remains a cornerstone in the study of regularity properties of solutions to PDEs. It is flexible enough to be carried out in the context of other divergence-form PDEs in doubling quasi-metric spaces that sustain a Poincaré-type inequality (for instance, see [1] for the $p(x)$-Laplacian, [4] for quasi-minimizers of $p$-Dirichlet integrals, [13] for degenerate elliptic PDEs, [16] for infinite graphs, [23] for Dirichlet forms in homogeneous spaces, [26] for $X$-elliptic operators, etc.).

The variational tools (e.g., Caccioppoli's inequality and other energy estimates) in the divergence-form context were replaced by measure-theoretic/probabilistic ones in the setting of non-divergence-form elliptic PDEs. Most notably, by the so-called *critical density property* for non-negative supersolutions: there exist $\varepsilon, \gamma \in (0,1)$ such that for $B \in \mathcal{B}_\Omega$

$$\left|\left\{x \in 2B\colon u(x) > 1\right\}\right| \geqslant \varepsilon|2B| \quad \Rightarrow \quad \inf_B u > \gamma. \tag{1.4}$$

Property (1.4) lies at the heart of the techniques developed by Krylov and Safonov [30,31] to prove Harnack's inequality for non-negative solutions to non-divergence-form elliptic and parabolic PDEs. In fact, this measure-theoretic approach, greatly simplified and enriched by L. Caffarelli in [9], appears to capture the essence of ellipticity (see Remark 6) in non-variational settings including fully non-linear elliptic PDEs [9] and degenerate elliptic PDEs such as the linearized Monge–Ampère equation [11,12] as well as variational ones including divergence-form elliptic PDEs with *a priori* energy estimates (see [17] in the Euclidean case and [29] for metric spaces with a calculus of order 1, that is, those metric spaces admitting Sobolev or Poincaré-type inequalities) and adjoint solutions to non-divergence elliptic operators [20], just to name a few. We mention that, in all rigor, the linearized Monge–Ampère equation possesses a double nature, variational and non-variational. It was precisely the pioneering work of Caffarelli and Gutiérrez on the linearized Monge–Ampère equation [11,12], where convex functions prescribe the relevant geometric and measure-theoretic framework, that led to the axiomatization of the Krylov–Safanov–Caffarelli approach in the context of doubling quasi-metric spaces [3,18,39]. Some notation is in order.

**Definition 1.** Let $(X, d, \mu)$ be a doubling quasi-metric space (see Section 2). Following [18], $\mathbb{K}_\Omega$ denotes a family of $\mu$-measurable functions with domain contained in an open set $\Omega$, and if $u \in \mathbb{K}_\Omega$ and $A \subset \mathrm{dom}(u)$ then we write $u \in \mathbb{K}_\Omega(A)$. Here $\mathrm{dom}(u)$ stands for the domain of the function $u$. We say that $\mathbb{K}_\Omega$ is closed under *multiplication by positive constants* if whenever $u \in K_\Omega$ and $\tau > 0$, then $\tau u \in \mathbb{K}_\Omega$. Also, we say that $\mathbb{K}_\Omega$ is closed under *multiplication by small constants* if whenever $u \in K_\Omega$ and $\tau \in (0,1)$, then $\tau u \in \mathbb{K}_\Omega$.