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Landesman–Lazer conditions at half-eigenvalues of the p-Laplacian ☆

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ABSTRACT

We study the existence of solutions of the Dirichlet problem

$$-\phi_p(u')' - a_+\phi_p(u^+) + a_-\phi_p(u^-) - \lambda\phi_p(u) = f(x,u),\quad x \in (0,1), \tag{1}$$

$$u(0) = u(1) = 0, \tag{2}$$

where $p > 1$, $\phi_p(s) := |s|^{p-1}\operatorname{sgn} s$ for $s \in \mathbb{R}$, the coefficients $a_\pm \in C^0[0,1]$, $\lambda \in \mathbb{R}$, and $u^\pm := \max\{\pm u, 0\}$. We suppose that $f \in C^1([0,1]\times\mathbb{R})$ and that there exist $f_\pm \in C^0[0,1]$ such that $\lim_{\xi\to\pm\infty} f(x,\xi) = f_\pm(x)$, for all $x \in [0,1]$. With these conditions the problem (1)–(2) is said to have a 'jumping nonlinearity'. We also suppose that the problem

$$-\phi_p(u')' = a_+\phi_p(u^+) - a_-\phi_p(u^-) + \lambda\phi_p(u) \quad \text{on } (0,1), \tag{3}$$

together with (2), has a non-trivial solution u. That is, λ is a 'half-eigenvalue' of (2)–(3), and the problem (1)–(2) is said to be 'resonant'. Combining a shooting method with so-called 'Landesman–Lazer' conditions, we show that the problem (1)–(2) has a solution.

Most previous existence results for jumping nonlinearity problems at resonance have considered the case where the coefficients $a_\pm$ are constants, and the resonance has been at a point in the 'Fučík spectrum'. Even in this constant coefficient case our result extends previous results. In particular, previous variational approaches have

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required strong conditions on the location of the resonant point, whereas our result applies to any point in the Fučík spectrum.

1. Introduction

We consider the p-Laplacian Dirichlet problem

$$-\phi_p\big(u'\big)' - a_+\phi_p\big(u^+\big) + a_-\phi_p\big(u^-\big) - \lambda\phi_p(u) = f(x,u), \quad x \in (0,1), \tag{1.1}$$

$$u(0) = u(1) = 0, \tag{1.2}$$

where $p > 1$, $\phi_p(s) := |s|^{p-1}\operatorname{sgn} s$ for $s \in \mathbb{R}$, the coefficients $a_\pm \in C^0[0,1]$, $\lambda \in \mathbb{R}$, and $u^\pm := \max\{\pm u, 0\}$. We assume that $f \in C^1([0,1]\times\mathbb{R})$ satisfies the following hypotheses:

$(f1)$ there exist $f_\pm \in C^0[0,1]$ such that

$$\lim_{\xi\to\pm\infty} f(x,\xi) = f_\pm(x), \quad x \in [0,1], \tag{1.3}$$

$(f2)$ there exist $K_1 > 0$ and $\rho \in [0,1)$, with $\rho > 2-p$ if $p \leqslant 2$, such that

$$\big|\xi^\rho f_\xi(x,\xi)\big| \leqslant K_1, \quad x \in [0,1],\ \xi \in \mathbb{R}. \tag{1.4}$$

Of course, it follows immediately from $(f1)$ that there exists $K_0 > 0$ such that

$$\big|f(x,\xi)\big| \leqslant K_0, \quad x \in [0,1],\ \xi \in \mathbb{R}. \tag{1.5}$$

These conditions imply that the asymptotic behaviour of (1.1) as $u \to \pm\infty$ is determined by the coefficients $a_\pm$ and $f_\pm$ (and the value of λ), and that these asymptotic behaviours may be different. Such problems are often termed *jumping*.

Let us now introduce some basic notations and definitions. The spaces $C^i[0,1]$ will be endowed with their usual sup-type norms $|\cdot|_i$, $i = 0,1$. We define D_p to be the set of functions $u \in C^1[0,1]$ such that $\phi_p(u') \in C^1[0,1]$, and an operator $\Delta_p : D(\Delta_p) \to C^0[0,1]$ by

$$D(\Delta_p) := \big\{u \in D_p\colon u \text{ satisfies } (1.2)\big\},$$

$$\Delta_p(u) := \phi_p\big(u'\big)', \quad u \in D(\Delta_p).$$

In addition, we denote by $u \to f(u)$ the Nemitskii mapping associated with f, that is, $f(u)$ is defined by $f(u)(x) := f(x,u(x))$, $x \in [0,1]$. (We will use a similar notation for other Nemitskii mappings.) Clearly, $u \to f(u)$ is a continuous mapping from $C^0[0,1]$ to $C^0[0,1]$. With this notation the problem (1.1)–(1.2) can now be rewritten as

$$-\Delta_p(u) - a_+\phi_p\big(u^+\big) + a_-\phi_p\big(u^-\big) - \lambda\phi_p(u) = f(u), \quad u \in D(\Delta_p). \tag{1.6}$$

Related to (1.6) is the half-eigenvalue problem

$$-\Delta_p(u) - a_+\phi_p\big(u^+\big) + a_-\phi_p\big(u^-\big) = \lambda\phi_p(u), \quad u \in D(\Delta_p). \tag{1.7}$$