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A well-posed Cauchy problem for an evolution equation with coefficients of low regularity

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ABSTRACT

In the hyperbolic Cauchy problem, the well-posedness in Sobolev spaces is strictly related to the modulus of continuity of the coefficients. This holds true for p-evolution equations with real characteristics ($p = 1$ hyperbolic equations, $p = 2$ vibrating plate and Schrödinger type models, ...). We show that, for $p \geqslant 2$, a lack of regularity in t can be balanced by a damping of the too fast oscillations as the space variable $x \to \infty$. This cannot happen in the hyperbolic case $p = 1$ because of the finite speed of propagation.

1. Introduction and main results

We deal with the model Cauchy problem

$$\begin{cases} D_t^2 u - a(t,x) D_x^{2p} u = 0 \\ u(0,x) = u_0(x), \qquad D_t u(0,x) = u_1(x) \end{cases} \tag{1.1}$$

where $(t,x) \in [0,T] \times \mathbb{R}_x$, $D = \frac{1}{i}\partial$ and the positive integer p represents the evolution degree. The coefficient $a(t,x)$ is at least such that $a \in C([0,T]; \mathcal{B}^\infty)$ with $\mathcal{B}^\infty = \mathcal{B}^\infty(\mathbb{R}_x)$ the space of all the functions of the variable x which are bounded together with all their derivatives. With some more technicalities, one can consider the equation $D_t^2 u - a(t,x)\Delta_x^p u = 0$ for $x \in \mathbb{R}^n$. We consider here the case $n = 1$ for sake of simplicity.

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We take Cauchy data in Sobolev spaces

$$u_0 \in H^s, \qquad u_1 \in H^{s-p}$$

for any fixed index s and look for solutions which remain in Sobolev spaces at any time t. Precisely, we say that problem (1.1) is L^2 well-posed if there exists a unique solution

$$u \in C\big([0,T]; H^s\big) \cap C^1\big([0,T]; H^{s-p}\big) \tag{1.2}$$

and we say that it is well-posed with a loss of derivatives if the unique solution is such that

$$u \in C\big([0,T]; H^{s-\delta}\big) \cap C^1\big([0,T]; H^{s-p-\delta}\big) \tag{1.3}$$

for some positive δ.

For any $p \geqslant 1$, from the Lax–Mizohata theorem we can have forward and backward well-posedness with Cauchy data at $t = t_0$ only if the roots of

$$\tau^2 - a(t,x)\xi^{2p}$$

are real at $t = t_0$. We assume

$$a(t,x) \geqslant a_0 > 0, \tag{1.4}$$

hence the characteristics are real and distinct at any (t,x). In particular, for $p = 1$ in (1.1) we have a strictly hyperbolic Cauchy problem for the wave equation

$$u_{tt} - a(t,x)u_{xx} = 0.$$

For $p = 2$ we are dealing with the beam model

$$u_{tt} + a(t,x)u_{xxxx} = 0$$

and with the related Schrödinger equation

$$\frac{1}{i}u_t \pm \sqrt{a(t,x)}u_{xx} = 0$$

by algebraic factorization of the symbol.

Starting from [10], we know that well-posedness and modulus of continuity of the coefficients are deeply connected in the hyperbolic Cauchy problem. For the wave equation

$$u_{tt} - a(t)u_{xx} = 0,$$

$a(t) \geqslant a_0 > 0$, the Lipschitz condition

$$\big|a(t+\tau) - a(t)\big| \leqslant L|\tau|$$

is sharp for L^2 well-posedness while the Log-Lipschitz regularity

$$\big|a(t+\tau) - a(t)\big| \leqslant L|\tau|\big|\log|\tau|\big|$$