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# A system of quadratic nonlinear Klein–Gordon equations in 2d

Nakao Hayashi[a,*], Pavel I. Naumkin[b]

[a] Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Toyonaka, 560-0043, Japan
[b] Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, Campus Morelia, AP 61-3 (Xangari), Morelia CP 58089, Michoacán, Mexico

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ABSTRACT

We consider the system of quadratic nonlinear Klein–Gordon equations

$$\begin{cases} \partial_t^2 v_1 - \Delta v_1 + m_1^2 v_1 = v_1 v_2, \\ \partial_t^2 v_2 - \Delta v_2 + m_2^2 v_2 = v_1^2 \end{cases}$$

in two space dimensions. Under the mass condition such that $2m_1 > m_2$, we construct the scattering operator in an almost natural weighted Sobolev class $\mathbf{H}^{1+\delta,1}$ or the scattering problem in lower order Sobolev class $\mathbf{H}^{1+\delta}_{\frac{4}{3}-\delta} \cap \mathbf{H}^{1+\delta}$, where $\delta > 0$ is small.



## 1. Introduction

We consider a system of quadratic nonlinear Klein–Gordon equations

$$\begin{cases} \partial_t^2 v_1 - \Delta v_1 + m_1^2 v_1 = v_1 v_2, \\ \partial_t^2 v_2 - \Delta v_2 + m_2^2 v_2 = v_1^2 \end{cases} \tag{1.1}$$

in two space dimensions, where $v_1, v_2$ are real-valued functions, $\Delta = \sum_{j=1}^2 \partial_j^2$ is the Laplacian, $\partial_j = \partial/\partial x_j$, $m_1, m_2 > 0$ are the masses of the particles. We have the conservation of the energy

* Corresponding author.
*E-mail addresses:* nhayashi@math.sci.osaka-u.ac.jp (N. Hayashi), pavelni@matmor.unam.mx (P.I. Naumkin).

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$$\frac{d}{dt}\left(E_1(t) + 2E_2(t) - \int_{\mathbf{R}^2} v_1^2 v_2(t,x)\,dx\right) = 0,$$

where the energy

$$E_j(t) = \left\|\partial_t v_j(t)\right\|_{\mathbf{L}^2}^2 + \left\|\nabla v_j(t)\right\|_{\mathbf{L}^2}^2 + m_j^2\left\|v_j(t)\right\|_{\mathbf{L}^2}^2.$$

However it is not enough to ensure the global existence of solutions.

Quadratic nonlinear Klein–Gordon equations in two space dimensions are critical for the scattering problem. In order to remove the critical nonlinearities, we use the method of the normal forms of Shatah [15] which requires us to estimate the bilinear operators depending on the nonlinearities and the Klein–Gordon evolution group $\mathcal{U}_m(t) = e^{-it\langle i\nabla\rangle_m}$.

It is shown in [16] that a small solution $(v_1, v_2)$ to the Cauchy problem for system (1.1) exists globally and is asymptotically free. Moreover, under the nonresonance mass condition $2m_1 \neq m_2$ and rather strong hypotheses on the initial data, it was proved that the time decay estimates for the solutions are the same as that for the linear problem. Global existence and time decay of small solutions were obtained in [10] for the resonance case $2m_1 = m_2$, under some regularity and compactness conditions on the initial data. However the large time asymptotics is not known for the case of $2m_1 = m_2$. Our purpose in this paper is twofold. First, under the mass condition $2m_1 > m_2$, we will prove the global existence of small solutions, find their asymptotic behavior and give a positive answer to the scattering problem in an almost natural weighted Sobolev space $\mathbf{H}^{1+\delta,1}(\mathbf{R}^2)$ with $\delta > 0$. Another purpose is to prove existence of the scattering states and wave operators in the lower order Sobolev space $\mathbf{H}^{1+\delta}_{\frac{4}{3}-\delta}(\mathbf{R}^2) \cap \mathbf{H}^{1+\delta}(\mathbf{R}^2)$. It seems that the method of algebraic normal forms used in [16] does not work well for the construction of the scattering operator even if we consider the problem in higher order Sobolev spaces. On the other hand, the method of algebraic normal forms works well for a proof of global existence of solutions in the case of $2m_1 < m_2$. However our proof depends on Proposition 2.1, and so does not work for this case. Thus the existence of the scattering operator is an open problem for the case of $2m_1 \leqslant m_2$.

The existence of the scattering states in the neighborhood of the origin in the space $\mathbf{H}^{1+\frac{n}{2},1}(\mathbf{R}^n) \cap \mathbf{H}^{\frac{n}{2},1}(\mathbf{R}^n)$ for the super critical nonlinear Klein–Gordon equation

$$\begin{cases} u_{tt} - \Delta u + u = \mu|u|^{p-1}u, & (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbf{R}^n, \end{cases}$$

was proved in [6], where $p > 1 + \frac{2}{n}$, $\mu \in \mathbf{C}$, $n = 1, 2$. The same method is useful for the system of super critical nonlinear Klein–Gordon equations

$$\begin{cases} \partial_t^2 v_1 - \Delta v_1 + m_1^2 v_1 = |v_2|^{p-2} v_2 v_1, \\ \partial_t^2 v_2 - \Delta v_2 + m_2^2 v_2 = |v_2|^{p-2} v_1^2. \end{cases}$$

Note that the mass condition is not needed in the super critical case $p > 1 + \frac{2}{n}$. The regularity of order $1 + \frac{n}{2}$ was required for the above problem, so that the method of normal forms of [15], which is one of our tools, works well to reduce the order of regularity of the data.

By changing the dependent variables $u_j = \frac{1}{2}(v_j + i\langle i\nabla\rangle_{m_j}^{-1}\partial_t v_j)$, we find that $u_1$ and $u_2$ satisfy the following system of equations

$$\begin{cases} \mathcal{L}_{m_1} u_1 = 2i\langle i\nabla\rangle_{m_1}^{-1}(\operatorname{Re} u_1)(\operatorname{Re} u_2), \\ \mathcal{L}_{m_2} u_2 = 2i\langle i\nabla\rangle_{m_2}^{-1}(\operatorname{Re} u_1)^2, \end{cases} \tag{1.2}$$

where $\mathcal{L}_m = \partial_t + i\langle i\nabla\rangle_m$, $\langle i\nabla\rangle_m = \sqrt{m^2 - \Delta}$.