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A generalized Pohožaev identity and uniqueness of positive radial solutions of $\Delta u + g(r)u + h(r)u^p = 0$

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This paper is dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

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ABSTRACT

We show a new generalized Pohožaev identity for $\Delta u + g(r)u + h(r)u^p = 0$, and we apply it to show the uniqueness of a positive radial solution of the equation in a ball, the entire space, an annulus, or an exterior domain under Dirichlet boundary condition.

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1. Introduction

We consider the semilinear elliptic equation

$$\Delta u(x) + g\bigl(|x|\bigr)u(x) + h\bigl(|x|\bigr)u(x)^p = 0 \tag{1.1}$$

in a ball in $\mathbb{R}^n$ with radius $R \in (0, \infty]$ under the Dirichlet boundary condition

$$\begin{cases} u(x) = 0 & \text{for } |x| = R \quad \text{in the case of } R < \infty, \\ u(x) \to 0 & \text{as } |x| \to \infty \quad \text{in the case of } R = \infty, \end{cases}$$

where $n \geqslant 2$, $p > 1$ and $g, h : (0, R) \to \mathbb{R}$ are appropriate functions. Problem (1.1) includes many important examples, like the scalar field equation

$$\Delta u(x) - u(x) + u(x)^p = 0 \quad \text{in } \mathbb{R}^n, \qquad u(x) \to 0 \quad \text{as } |x| \to \infty, \tag{1.2}$$

where $n \geqslant 3$ and $1 < p < (n+2)/(n-2)$. The problem has a long history. Coffman [14] first obtained the uniqueness result in the case $n = 3$ and $p = 3$. Later Peletier–Serrin [53], McLeod–Serrin [45] and McLeod [44] generalized Coffman's result, and finally, Kwong [35] established the uniqueness of a positive solution of (1.2) up to translation.

Since we consider a positive radial solution of (1.1), we study the uniqueness of a positive solution of

$$\begin{cases} u_{rr}(r) + \dfrac{n-1}{r}u_r + g(r)u(r) + h(r)u(r)^p = 0, \quad 0 < r < R, \\ u(0) \in (0, \infty), \quad u(R) = 0. \end{cases} \tag{1.3}$$

Here, in the case of $R = \infty$, $u(R) = 0$ means $u(r) \to 0$ as $r \to \infty$. Such a problem has been studied by many researchers; see [2,10,13,14,18,19,28–35,37,44,45,47–49,53,54,58,63,66–70] and others. In many of them, Pohožaev type identities were used to establish their uniqueness results. In particular, Yanagida [66] used only a Pohožaev type identity to obtain his uniqueness results. Although his results are applicable to many problems, it is not easy to see how he found his Pohožaev identity.

In this paper, we introduce a new generalized Pohožaev identity, and by using it only, we give uniqueness theorems for (1.3). The concept of our Pohožaev identity is very clear and it is easy to see how we find it. Our proofs for the uniqueness results are also clear and our results are applicable to various problems; see Section 5. We also study the annular domain problem

$$\begin{cases} u_{rr}(r) + \dfrac{n-1}{r}u_r + g(r)u(r) + h(r)u(r)^p = 0, \quad R' < r < R, \\ u\bigl(R'\bigr) = 0, \quad u(R) = 0, \end{cases} \tag{1.4}$$

where $n \geqslant 1$, $p > 1$, $0 < R' < R \leqslant \infty$ and $g, h : (R', R) \to \mathbb{R}$. Such kinds of results were studied in [12,15,16,21,22,36,37,48,58,63,64] and others. The uniqueness of a positive radial solution of

$$\Delta u(x) - u(x) + u(x)^p = 0 \quad \text{for } R' < |x| < R, \qquad u(x) = 0 \quad \text{for } |x| = R', R \tag{1.5}$$

has a different story. After the contributions of Coffman [15], Yadava [63,64], Kwong–Zhang [36], Kwong–Li [37] and others, Tang [58] and Felmer–Martínez–Tanaka [21] established the uniqueness of a positive radial solution. We also apply our Pohožaev identity to problem (1.4) and we give uniqueness results of a positive solution. More generally, instead of (1.3) or (1.4), we consider the problem