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Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$

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ABSTRACT

We study the existence and multiplicity of solutions for elliptic equations in $\mathbb{R}^N$, driven by a non-local integro-differential operator, which main prototype is the fractional Laplacian. The model under consideration, denoted by $(\mathscr{P}_\lambda)$, depends on a real parameter λ and involves two superlinear nonlinearities, one of which could be critical or even supercritical. The main theorem of the paper establishes the existence of three critical values of λ which divide the real line in different intervals, where $(\mathscr{P}_\lambda)$ admits no solutions, at least one nontrivial non-negative entire solution and two nontrivial non-negative entire solutions.

1. Introduction

In this paper we prove the existence and multiplicity of solutions for non-local integro-differential equations in $\mathbb{R}^N$, whose prototype is given by

$$(-\Delta)^s u + a(x)u = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N, \qquad (\mathscr{P}_\lambda)$$

where $\lambda \in \mathbb{R}$, $0 < s < 1$, $2s < N$ and $(-\Delta)^s$ is the fractional Laplacian operator. Up to normalization factors, $(-\Delta)^s u$ is defined pointwise for x in $\mathbb{R}^N$ by

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$$(-\Delta)^s u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\,dy,$$

along any rapidly decaying function u of class $C^\infty(\mathbb{R}^N)$, see Lemma 3.5 of [18].

The nonlinear terms in $(\mathscr{P}_\lambda)$ *are related to the main elliptic part by* the request that

$$2 < q < \min\{r, 2^*\}, \tag{1.1}$$

where $2^* = 2N/(N-2s)$ is the critical Sobolev exponent for $H^s(\mathbb{R}^N)$. *The coefficient a is supposed to be in* $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ *and to satisfy for a.a.* $x \in \mathbb{R}^N$

$$\nu(x) = \max\{a(x), (1+|x|)^{-2s}\}, \quad a(x) \geqslant \kappa\,\nu(x), \tag{1.2}$$

for some constant $\kappa \in (0,1]$. *The weight w verifies*

$$w \in L^{\wp}(\mathbb{R}^N) \cap L^\sigma_{\mathrm{loc}}(\mathbb{R}^N), \quad \text{with } \wp = 2^*/(2^*-q), \quad \sigma > \wp, \tag{1.3}$$

while *h is a positive weight of class* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Finally, *h and w are related by the condition*

$$\int_{\mathbb{R}^N} \left[\frac{w(x)^r}{h(x)^q}\right]^{1/(r-q)} dx = H \in \mathbb{R}^+. \tag{1.4}$$

The main result of the paper is

Theorem 1.1. *Under the above assumptions there exist* λ^*, λ^{**} *and* $\bar\lambda$, *with* $0 < \lambda^* \leqslant \lambda^{**} \leqslant \bar\lambda$ *such that Eq.* $(\mathscr{P}_\lambda)$ *admits*

(i) *only the trivial solution if* $\lambda < \lambda^*$;
(ii) *a nontrivial non-negative entire solution if and only if* $\lambda \geqslant \lambda^{**}$;
(iii) *at least two nontrivial non-negative entire solutions if* $\lambda > \bar\lambda$.

The definition of *entire solution* for $(\mathscr{P}_\lambda)$, as well as the proof of Theorem 1.1(i), are given in Section 2, after the introduction of the main solution space X. Some preliminary results for existence are presented in Section 3 and in Appendix A. The proof of Theorem 1.1(ii) is discussed in Section 4, while Theorem 1.1(iii) is proved in Section 5.

For standing wave solutions of fractional Schrödinger equations in $\mathbb{R}^N$ we refer to [20,22,32,13, 28], [19, Section 5] and to the references therein. Models governed by unbounded potentials V are investigated in [14] and in its recent extension [27]. All these papers, however, deal with problems which are not directly comparable to $(\mathscr{P}_\lambda)$. The present work is more related to the results on general quasilinear elliptic problems given in [4]. Indeed, in [4], as a corollary of the main theorems, we proved under (1.4) that there exists $\lambda^* > 0$ such that

$$\begin{gathered} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + a(x)|u|^{p-2}u = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N, \\ 1 < p < N, \qquad \max\{2,p\} < q < \min\{r, p^*\}, \qquad p^* = \frac{Np}{N-p}, \end{gathered} \tag{$\mathscr{E}_\lambda$}$$

admits at least a nontrivial non-negative entire solution if and only if $\lambda \geqslant \lambda^*$. Theorem 1.1(ii) extends Theorem A of [4] to non-local integro-differential equations. It would be interesting to understand if $\lambda^* = \lambda^{**}$ in Theorem 1.1. This possible gap does not rise in [4]. Indeed, if u is a solution of $(\mathscr{E}_\lambda)$ also $|u|$ is. The situation is more delicate for $(\mathscr{P}_\lambda)$, since the fractional Laplacian itself does not guarantee