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Boundary estimates for solutions to operators of p-Laplace type with lower order terms

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ABSTRACT

In this paper we study the boundary behavior of solutions to equations of the form

$$\nabla \cdot A(x, \nabla u) + B(x, \nabla u) = 0,$$

in a domain $\Omega \subset \mathbf{R}^n$, assuming that Ω is a δ-Reifenberg flat domain for δ sufficiently small. The function A is assumed to be of p-Laplace character. Concerning B, we assume that $|\nabla_\eta B(x, \eta)| \leqslant c|\eta|^{p-2}$, $|B(x, \eta)| \leqslant c|\eta|^{p-1}$, for some constant c, and that $B(x, \eta) = |\eta|^{p-1} B(x, \eta/|\eta|)$, whenever $x \in \mathbf{R}^n$, $\eta \in \mathbf{R}^n \setminus \{0\}$. In particular, we generalize the results proved in J. Lewis et al. (2008) [12] concerning the equation $\nabla \cdot A(x, \nabla u) = 0$, to equations including lower order terms.

1. Introduction

In [13,14] a number of results concerning the boundary behavior of positive p-harmonic functions, $1 < p < \infty$, in a bounded Lipschitz domain $\Omega \subset \mathbf{R}^n$ were proved. In particular, a boundary Harnack inequality, as well as the Hölder continuity for ratios of positive p-harmonic functions, $1 < p < \infty$, vanishing on a portion of $\partial\Omega$ were established. Furthermore, the p-Martin boundary problem at $w \in \partial\Omega$ was resolved under the assumption that Ω is either convex, C^1-regular or a Lipschitz domain with small constant. Also, in [15] these questions were resolved for p-harmonic functions vanishing

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on a portion of certain Reifenberg flat and Ahlfors regular NTA-domains. In [12], the second and third author, together with John Lewis, began the generalization of the result in [13–15] to more general operators of p-Laplace type allowing for variable coefficients. In particular, in [12], new results concerning boundary Harnack inequalities and the Martin boundary problem, in Reifenberg flat domains, for operators of p-Laplace type of the form $\nabla \cdot A(x, \nabla u) = 0$ were established. The purpose of this paper is to take the analysis in [12] one step further by establishing the corresponding results for operators of the form $\nabla \cdot A(x, \nabla u) + B(x, \nabla u) = 0$, i.e., we here allow for lower order terms. From a technical point of view, several of the estimates proved in this paper are proved, as outlined in the bulk of the paper, by scaling arguments and by perturbing off the corresponding results in [12]. As a general motivation for this study and for the generalization of boundary Harnack inequalities to p-Laplace operators 'as general as possible' we mention the importance of these type of results to the study of free boundary problems. In particular, we refer to [16–20] where several problems of free boundary character for the p-Laplace operator are resolved.

To state our results we need to introduce some notation. Points in the Euclidean n-space $\mathbf{R}^n$ are denoted by $x = (x_1, \dots, x_n)$ or (x', x_n) where $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. Let $\bar{E}$, ∂E, diam E, be the closure, boundary, diameter, of the set $E \subset \mathbf{R}^n$ and let $d(y, E)$ equal the distance from $y \in \mathbf{R}^n$ to E. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^n$ and $|x| = \langle x, x \rangle^{1/2}$ is the Euclidean norm of x. Put $B(x, r) = \{y \in \mathbf{R}^n\colon\ |x - y| < r\}$ whenever $x \in \mathbf{R}^n$, $r > 0$, and let dx be the Lebesgue n-measure on $\mathbf{R}^n$. We let

$$h(E, F) = \max\bigl(\sup\bigl\{d(y, E)\colon\ y \in F\bigr\}, \sup\bigl\{d(y, F)\colon\ y \in E\bigr\}\bigr)$$

be the Hausdorff distance between the sets $E, F \subset \mathbf{R}^n$. If $O \subset \mathbf{R}^n$ is open and $1 \leqslant q \leqslant \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are qth power integrable on O. Let $\|f\|_{1,q} = \|f\|_q + \||\nabla f|\|_q$ be the norm in $W^{1,q}(O)$ where $\|\cdot\|_q$ denotes the usual Lebesgue q-norm in O. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. By $\nabla\cdot$ we denote the divergence operator.

We now introduce the operators of p-Laplace type which we consider in this paper.

Definition 1.1. Let $p, \alpha_1, \alpha_2, \beta_1, \beta_2 \in (1, \infty)$ and $\gamma \in (0, 1]$. Let $A = (A_1, \dots, A_n) : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, $B : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, assume that $A = A(x, \eta)$, $B = B(x, \eta)$ are continuous on $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ and that A, B, for fixed $x \in \mathbf{R}^n$, are continuously differentiable in η_k, for every $k \in \{1, \dots, n\}$, whenever $\eta \in \mathbf{R}^n \setminus \{0\}$. We say that the pair (A, B) belongs to the class $M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma)$, $(A, B) \in M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma)$ for short, if the following conditions are satisfied for $j \in \{1, \dots, n\}$ whenever x, y, $\xi \in \mathbf{R}^n$ and $\eta \in \mathbf{R}^n \setminus \{0\}$:

$$(\mathrm{i}_A) \quad \alpha_1^{-1} |\eta|^{p-2} |\xi|^2 \leqslant \sum_{i,j=1}^{n} \frac{\partial A_i}{\partial \eta_j}(x, \eta) \xi_i \xi_j \leqslant \alpha_1 |\eta|^{p-2} |\xi|^2,$$

$$(\mathrm{ii}_A) \quad \bigl|A(x, \eta) - A(y, \eta)\bigr| \leqslant \alpha_2 |x - y|^\gamma |\eta|^{p-1},$$

$$(\mathrm{iii}_A) \quad A(x, \eta) = |\eta|^{p-1} A\bigl(x, \eta/|\eta|\bigr),$$

$$(\mathrm{i}_B) \quad \left|\frac{\partial B}{\partial \eta_j}(x, \eta)\right| \leqslant \beta_1 |\eta|^{p-2},$$

$$(\mathrm{ii}_B) \quad \bigl|B(x, \eta)\bigr| \leqslant \beta_2 |\eta|^{p-1},$$

$$(\mathrm{iii}_B) \quad B(x, \eta) = |\eta|^{p-1} B\bigl(x, \eta/|\eta|\bigr).$$

If $\beta_2 \equiv 0$ we write, for short, $(A, 0) \in M_p(\alpha_1, \alpha_2, \gamma)$.