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Global well-posedness for a modified dissipative surface quasi-geostrophic equation in the critical Sobolev space H^1

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ABSTRACT

In this paper, we consider the following modified quasi-geostrophic equations

$$\partial_t\theta + \Lambda^\alpha\theta + u\vec{\nabla}\theta = 0, \qquad u = \Lambda^{\alpha-1}\mathcal{R}^\perp(\theta) \qquad \text{(MQG)}$$

where $\alpha \in]0,1[$ is a fixed parameter. This equation was recently introduced by P. Constantin, G. Iyer and J. Wu (2001) in [4] as a modification of the classical quasi-geostrophic equation. In this paper, we prove that for any initial data θ_* in the Sobolev space $H^1(\mathbb{R}^2)$, Eq. (MQG) has a global and smooth solution θ in $C(\mathbb{R}^+, H^1(\mathbb{R}^2))$.

1. Introduction and statement of results

In this paper, we are concerned with the modified dissipative surface quasi-geostrophic equation:

$$\begin{cases} \partial_t\theta + \Lambda^\alpha\theta + u\vec{\nabla}\theta = 0, \\ u = \Lambda^{\alpha-1}\mathcal{R}^\perp(\theta), \\ \theta|_{t=0} = \theta_*. \end{cases} \qquad \text{(MQG)}$$

Here, $\alpha \in]0,1[$ is a *fixed* real number, the unknown $\theta = \theta(t,x)$ is a real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}^2$, θ_* is a given initial data, $\mathcal{R}^\perp$ is the operator defined via Riesz transforms by

$$\mathcal{R}^\perp(\theta) = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta),$$

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and Λ^γ is the non-local operator defined through the Fourier transform by

$$\widehat{\Lambda^\gamma f}(\xi) = |\xi|^\gamma \hat{f}(\xi).$$

The equation (MQG) was recently introduced in [4] by P. Constantin, G. Iyer and J. Wu as a modification of the dissipative surface quasi-geostrophic equation

$$\begin{cases} \partial_t\theta + \Lambda^\alpha\theta + u\vec{\nabla}\theta = 0, \\ u = \mathcal{R}^\perp(\theta), \\ \theta|_{t=0} = \theta_*. \end{cases} \tag{QG}$$

By using the classical method of Galerkin approximations or by following the mollification method employed, for instance, by J. Leray in [8] for solving the Navier–Stokes equations (see also Chapter 11 of [7]), one can easily show that for any initial data θ_* in $L^2(\mathbb{R}^2)$ Eq. (MQG) has at least one weak solution θ belonging to the space

$$L^\infty\big(\mathbb{R}^+, L^2\big(\mathbb{R}^2\big)\big) \cap L^2_{\mathrm{loc}}\big(\mathbb{R}^+, H^{\alpha/2}\big(\mathbb{R}^2\big)\big)$$

and satisfying

$$\theta(t) \to \theta_* \quad \text{in } L^2\big(\mathbb{R}^2\big) \text{ as } t \to 0^+.$$

Such a solution θ is called a Leray–Hopf solution to the (MQG) equation.

In [4], the authors proved that any Leray–Hopf solution to Eq. (MQG) belongs to the space $C^\infty(]0, +\infty[\times \mathbb{R}^2)$.

Very recently, C. Miao and L. Xue in [9] have proved that for any initial data $\theta_* \in H^m(\mathbb{R}^2)$, with $m \in \mathbb{N}$ and $m > 2$, there exists a unique global solution $\theta \in C(\mathbb{R}^+, H^m(\mathbb{R}^2))$ to Eq. (MQG). Moreover, the solution θ satisfies the following regularity property:

$$\forall\gamma \geqslant 0, \quad t^\gamma\theta \in L^\infty\big(\mathbb{R}^+, H^{m+\gamma\alpha}\big(\mathbb{R}^2\big)\big).$$

One of the main properties of Eq. (MQG) is the following scaling invariance property: If θ is a solution to (MQG) with initial data θ_* then, for all $\lambda > 0$, the function $\theta_\lambda \equiv \theta(\lambda^\alpha t, \lambda x)$ is a solution to (MQG) with initial data $\theta_*^\lambda \equiv \theta_*(\lambda x)$. This leads us to introduce the notion of critical space: a functional space X is called a critical space for Eq. (MQG) if for all $f \in X$ and $\lambda > 0$, we have

$$\big\|f(\lambda.)\big\|_X = \|f\|_X.$$

For instance, the homogeneous Sobolev's space $\dot{H}^\sigma(\mathbb{R}^2)$ is a critical space if and only if $\sigma = 1$. Therefore, following the classical approach of Fujita and Kato [6], it is natural to ask whether it is true that Eq. (MQG) is globally well-posed for initial data in the critical space $\dot{H}^1(\mathbb{R}^2)$. In this paper, we give a slightly weaker result. In fact, we prove that for any initial data θ_* in the inhomogeneous Sobolev's space $H^1(\mathbb{R}^2)$, Eq. (MQG) has a global solution θ in $C([0, \infty[, H^1(\mathbb{R}^2))$. Precisely, we will prove the following result.

Theorem 1.1. *Let $\theta_* \in H^\sigma(\mathbb{R}^2)$ with $\sigma \geqslant 1$. Then there is a unique solution θ in*

$$C^1\big(\mathbb{R}^+, L^2\big(\mathbb{R}^2\big)\big) \cap C\big(\mathbb{R}^+, H^\sigma\big(\mathbb{R}^2\big)\big) \cap L^2_{\mathrm{loc}}\big(\mathbb{R}^+, H^{\sigma+\frac{\alpha}{2}}\big(\mathbb{R}^2\big)\big)$$