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Spectral properties of the linearized compressible Navier–Stokes equation around time-periodic parallel flow

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ABSTRACT

The linearized problem around a time-periodic parallel flow of the compressible Navier–Stokes equation in an infinite layer is investigated. By using the Floquet theory, spectral properties of the evolution operator associated with the linearized problem are studied in detail. The Floquet representation of a low frequency part of the evolution operator, which plays an important role in the study of the nonlinear problem, is obtained.

1. Introduction

In this paper we study spectral properties of the linearized operator around a time-periodic solution to the compressible Navier–Stokes equation with time-periodic external force and time-periodic boundary condition.

We consider the system of equations

$$\partial_{\tilde{t}}\tilde{\rho} + \operatorname{div}(\tilde{\rho}\tilde{v}) = 0, \tag{1.1}$$

$$\tilde{\rho}(\partial_{\tilde{t}}\tilde{v} + \tilde{v}\cdot\nabla\tilde{v}) - \mu\Delta\tilde{v} - (\mu+\mu')\nabla\operatorname{div}\tilde{v} + \nabla\tilde{P}(\tilde{\rho}) = \tilde{\rho}\tilde{\mathbf{g}}, \tag{1.2}$$

in an n-dimensional infinite layer $\Omega_\ell = \mathbb{R}^{n-1}\times(0,\ell)$:

$$\Omega_\ell = \{\tilde{x} = {}^T(\tilde{x}', \tilde{x}_n);\ \tilde{x}' = {}^T(\tilde{x}_1,\dots,\tilde{x}_{n-1}) \in \mathbb{R}^{n-1},\ 0 < \tilde{x}_n < \ell\}.$$

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Here, $n \geqslant 2$; $\widetilde{\rho} = \widetilde{\rho}(\widetilde{x}, \widetilde{t})$ and $\widetilde{v} = {}^T(\widetilde{v}^1(\widetilde{x}, \widetilde{t}), \ldots, \widetilde{v}^n(\widetilde{x}, \widetilde{t}))$ denote the unknown density and velocity at time $\widetilde{t} \geqslant 0$ and position $\widetilde{x} \in \Omega_\ell$, respectively; $\widetilde{P}$ is the pressure, smooth function of $\widetilde{\rho}$, where for given $\rho_* > 0$ we assume $\widetilde{P}'(\rho_*) > 0$; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying $\mu > 0$, $\frac{2}{n}\mu + \mu' \geqslant 0$; div, ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to $\widetilde{x}$. Here and in what follows ${}^T\cdot$ denotes the transposition.

In (1.2) $\widetilde{\boldsymbol{g}}$ is assumed to have the form

$$\widetilde{\boldsymbol{g}} = {}^T\big(\widetilde{g}^1(\widetilde{x}_n, \widetilde{t}), 0, \ldots, 0, \widetilde{g}^n(\widetilde{x}_n)\big),$$

with $\widetilde{g}^1$ being a $\widetilde{T}$-periodic function in time, where $\widetilde{T} > 0$.

The system (1.1)–(1.2) is considered under boundary condition

$$\widetilde{v}|_{\widetilde{x}_n=0} = \widetilde{V}^1(t)\boldsymbol{e}_1, \qquad \widetilde{v}|_{\widetilde{x}_n=\ell} = 0, \tag{1.3}$$

and initial condition

$$(\widetilde{\rho}, \widetilde{v})|_{\widetilde{t}=0} = (\widetilde{\rho}_0, \widetilde{v}_0), \tag{1.4}$$

where $\widetilde{V}^1$ is a $\widetilde{T}$-periodic function of time and $\boldsymbol{e}_1 = {}^T(1, 0, \ldots, 0) \in \mathbb{R}^n$.

Under suitable conditions on $\widetilde{\boldsymbol{g}}$ and $\widetilde{V}^1$, problem (1.1)–(1.3) has smooth time-periodic solution $\bar{u}_p = {}^T(\bar{\rho}_p, \bar{v}_p)$ satisfying

$$\bar{\rho}_p = \bar{\rho}_p(\widetilde{x}_n) \geqslant \underline{\widetilde{\rho}}, \qquad \frac{1}{\ell}\int_0^\ell \bar{\rho}_p(\widetilde{x}_n)\, d\widetilde{x}_n = \rho_*,$$

$$\bar{v}_p = {}^T\big(\bar{v}_p^1(\widetilde{x}_n, \widetilde{t}), 0, \ldots, 0\big), \qquad \bar{v}_p^1(\widetilde{x}_n, \widetilde{t} + \widetilde{T}) = \bar{v}_p^1(\widetilde{x}_n, \widetilde{t}),$$

for a positive constant $\underline{\widetilde{\rho}}$.

Our main concern is asymptotic description of large time behavior of perturbations from $\bar{u}_p$ when Reynolds and Mach numbers are sufficiently small. For this purpose we consider the linearized problem in this paper.

To formulate the problem for perturbations, we introduce the following dimensionless variables:

$$\widetilde{x} = \ell x, \qquad \widetilde{t} = \frac{\ell}{V}t, \qquad \widetilde{v} = Vv, \qquad \widetilde{\rho} = \rho_*\rho, \qquad \widetilde{P} = \rho_* V^2 P,$$

with

$$\widetilde{w} = Vw, \qquad \widetilde{\phi} = \rho_*\gamma^{-2}\phi, \qquad \widetilde{V}^1 = VV^1, \qquad \widetilde{\boldsymbol{g}} = \frac{\mu V}{\rho_*\ell^2}\boldsymbol{g},$$

where

$$\gamma = \frac{\sqrt{\widetilde{P}'(\rho_*)}}{V}, \qquad V = \frac{\rho_*\ell^2}{\mu}\big\{\big|\partial_{\widetilde{t}}\widetilde{V}^1\big|_{C(\mathbb{R})} + \big|\widetilde{g}^1\big|_{C(\mathbb{R}\times[0,\ell])}\big\} + \big|\widetilde{V}^1\big|_{C(\mathbb{R})}.$$

In this paper we assume $V > 0$. Under this change of variables the domain Ω_ℓ is transformed into $\Omega = \mathbb{R}^{n-1} \times (0, 1)$ and $g^1(x_n, t)$, $V^1(t)$ are periodic in t with period $T > 0$ defined by

$$T = \frac{V}{\ell}\widetilde{T}.$$