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Singular limits for a parabolic–elliptic regularization of scalar conservation laws

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A B S T R A C T

We consider scalar hyperbolic conservation laws with a non-convex flux, in one space dimension. Then, weak solutions of the associated initial value problems can contain undercompressive shock waves. We regularize the hyperbolic equation by a parabolic–elliptic system that produces undercompressive waves in the hyperbolic limit regime. Moreover we show that in another limit regime, called capillarity limit, we recover solutions of a diffusive–dispersive regularization, which is the standard regularization used to approximate undercompressive waves. In fact the new parabolic–elliptic system can be understood as a low-order approximation of the third-order diffusive–dispersive regularization, thus sharing some similarities with the relaxation approximations. A study of the traveling waves for the parabolic–elliptic system completes the paper.

1. Introduction

Consider for the unknown $u = u(x,t)$ the homogeneous scalar law

$$u_t + f(u)_x = 0 \tag{1.1}$$

in $\Omega_T := \mathbb{R} \times (0,T)$, $T > 0$. Here, $f : \mathbb{R} \to \mathbb{R}$ is a smooth flux function which we assume to satisfy $f(0) = 0$ without any loss of generality. Provided that f is nonlinear, it is well known that solutions of the Cauchy problem associated to (1.1) can contain discontinuities, even for smooth initial data

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[7,28]; such solutions then must be understood in a weak sense. Interesting wave patterns occur if the flux f is not convex or concave, i.e., if f'' vanishes at one point at least: in this case it is possible to construct weak solutions that contain *undercompressive waves*.

In order to clarify what we mean by an undercompressive shock wave, consider for $u_\pm \in \mathbb{R}$ and $s = (f(u_+) - f(u_-))/(u_+ - u_-)$ the weak solution

$$U(x,t) = \begin{cases} u_- & \text{if } x - st < 0, \\ u_+ & \text{if } x - st > 0. \end{cases} \tag{1.2}$$

The function U is called a (compressive) Lax shock wave if the inequalities

$$f'(u_-) > s > f'(u_+) \tag{1.3}$$

hold. On the contrary, in this paper we focus on undercompressive shock waves U, which must fulfill either

$$f'(u_\pm) < s \quad \text{or} \quad f'(u_\pm) > s. \tag{1.4}$$

The remaining doubly undercompressive case $f'(u_-) < s < f'(u_+)$ is not taken into account in our study; this case occurs, for instance, in the Chapman–Jouguet theory of deflagration waves [6,11]. The major interest in undercompressive waves stems from the fact that they appear in several applications as bulk interfaces representing, e.g., phase boundaries [14], saturation fronts [32], precursors in thin film flow [2] and so on. In this framework the scalar case (1.1) considered in this paper must be understood as a simplified model which, however, captures main features of the problem.

In the general theory of conservation laws, a common approach to select meaningful weak solutions consists first in regularizing the system under consideration and then in studying the limit of the solutions obtained for the regularized system when some characteristic parameter vanishes. In the context of this paper the latter step is called the *sharp-interface* (or *hyperbolic*) limit. For a standard viscous regularization of (1.1) only compressive waves can occur in this limit [7, §8.6]. However, undercompressive waves can be driven as well by more refined regularizations. The diffusive–dispersive regularization

$$u_t^\varepsilon + f\big(u^\varepsilon\big)_x = \varepsilon u_{xx}^\varepsilon + \gamma \varepsilon^2 u_{xxx}^\varepsilon \tag{1.5}$$

is by now classical. Here, the singular parameter ε is assumed to be positive. The parameter $\gamma > 0$ keeps dissipation and dispersion in balance; it plays a role only in the study of traveling waves and then the dependence on γ is omitted in the following. For the analysis of the sharp-interface limit $\varepsilon \to 0$ of (1.5) we refer to [27,12]. A complete study of the traveling waves in the case $f(u) = u^3$, including (1.2) under the condition (1.4), can be found in [14]. Moreover, the analysis in [14] rules out the possibility of the doubly undercompressive case quoted above.

Let f be chosen such that f' is nonnegative or bounded. Denote by $F : \mathbb{R} \to \mathbb{R}$ the primitive of the flux f with $F(0) = 0$. Then it is readily checked that smooth solutions u^ε of (1.5) satisfy

$$\frac{d}{dt} E^\varepsilon\big[u^\varepsilon(.,t)\big] \leqslant 0 \tag{1.6}$$

for $t \in (0,T)$, where E^ε is the van der Waals' type energy

$$E^\varepsilon[u] = \int_{\mathbb{R}} \left(F(u) + \gamma \frac{\varepsilon^2}{2} u_x^2 \right) \mathrm{d}x. \tag{1.7}$$