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Mean-square random dynamical systems ☆

Peter E. Kloeden *, Thomas Lorenz

Institut für Mathematik, Goethe Universität, D-60054 Frankfurt am Main, Germany

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ABSTRACT

The classical theory of random dynamical systems is a pathwise theory based on a skew-product system consisting of a measure theoretic autonomous system that represents the driving noise and a topological cocycle mapping for the state evolution. This theory does not, however, apply to nonlocal dynamics such as when the dynamics of a sample path depends on other sample paths through an expectation or when the evolution of random sets depends on nonlocal properties such as the diameter of the sets. The authors showed recently in terms of stochastic morphological evolution equations that such nonlocal random dynamics can be characterized by a deterministic two-parameter process from the theory of nonautonomous dynamical systems acting on a state space of random variables or random sets with the mean-square topology. This observation is exploited here to provide a definition of mean-square random dynamical systems and their attractors. The main difficulty in applying the theory is the lack of useful characterizations of compact sets of mean-square random variables. It is illustrated through simple but instructive examples how this can be avoided in strictly contractive cases or circumvented by using weak compactness. The existence of a pullback attractor then follows from the much more easily determined mean-square ultimate boundedness of solutions.

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* Corresponding author.
E-mail addresses: kloeden@math.uni-frankfurt.de (P.E. Kloeden), lorenz@math.uni-frankfurt.de (T. Lorenz).

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1. Introduction

The solution of a nonautonomous differential equation on $\mathbb{R}^d$ depends on both the actual time t and the initial time t_0 rather than just on the elapsed time $t - t_0$ as in an autonomous system. The solution mapping $\phi(t, t_0, x_0)$ of an initial value problem for which an existence and uniqueness theorem holds then satisfies the *initial value* property, $\phi(t_0, t_0, x_0) = x_0$, the *two-parameter semigroup* evolution property

$$\phi(t_2, t_0, x_0) = \phi\big(t_2, t_1, \phi(t_1, t_0, x_0)\big), \quad t_0 \leqslant t_1 \leqslant t_2,$$

as well as the *continuity* property, $(t, t_0, x_0) \mapsto \phi(t, t_0, x_0)$ is continuous on the state space $\mathbb{R}^d$.

These properties of the solution mapping can be used to define an abstract nonautonomous dynamical system. Dafermos [6] and Hale [8] called such an abstract nonautonomous dynamical system a *process*. The asymptotic behavior of processes, in particular their pullback attractors, is investigated in the monograph [14].

This term "process" is used here in a deterministic context and is not to be confused with a stochastic process. Nevertheless, the solution mappings of stochastic differential equations on $\mathbb{R}^d$ also define a *deterministic* two-parameter semigroup or process in the mean-square sense generalized to allow for time-dependent domains of mean-square adapted random variables in $\mathbb{R}^d$. A very early version of this idea in a set-valued context can be found in [9] and more recently in [12]. The theory presented here differs from that of random dynamical systems developed by Arnold [1] and others (e.g. [3,4,7]), which involves a cocycle state mapping driven by stochastic process and pathwise convergence.

The definition of a mean-square random dynamical system is given in the next section together with definitions of forward and pullback attractors, and a theorem on the existence of pullback attractors based on that of a pullback absorbing family is adapted from the literature. The application of this theorem involves showing the mean-square ultimate boundedness of solutions and compactness conditions in a space of mean-square random variables. The first is now a standard task [15,18], but the second is difficult due to the lack of usable criteria for such compactness. Instead this difficulty is circumvented by restricting to uniformly contracting processes in Section 3 and to more easily handled weak compactness in Section 5. Examples of stochastic differential equations with a drift term satisfying a dissipative one-sided Lipschitz condition are given in Section 3, one a classical SDE and the other a nonlocal SDE with the expectation of the solution in a coefficient. Their attractors consist of singleton sets, i.e., a single entire solution, which is attracting in both the forward and pullback senses. In Section 4 the pullback asymptotic compactness of the process is established via total boundedness for a nonlocal SDE with nonadditive noise. Finally, in Section 5 a stochastic parabolic differential equation is considered and the existence of a weakly compact pullback attractor is established.

2. Mean-square random dynamical systems

Let $\mathbb{T} = \mathbb{Z}$ (discrete time case) or $\mathbb{R}$ (continuous time case) and define

$$\mathbb{T}^2_{\geqslant} := \big\{(t, t_0) \in \mathbb{T}^2 \colon t \geqslant t_0\big\}.$$

In addition, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{T}}, \mathbb{P})$ be a complete filtered probability space satisfying the usual hypothesis, i.e., $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ is an increasing and right continuous family of σ-sub-algebras of $\mathcal{F}$, which contains all $\mathbb{P}$-null sets. Essentially, $\mathcal{F}_t$ represents the information about the randomness at time t.

Finally, define $\mathfrak{X} := L^2(\Omega, \mathcal{F}; \mathbb{R}^d)$ and $\mathfrak{X}_t := L^2(\Omega, \mathcal{F}_t; \mathbb{R}^d)$ for each $t \in \mathbb{T}$.

Definition 1. A *mean-square random dynamical system* ϕ on the underlying space $\mathbb{R}^d$ with a probability set-up $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ is a family of mappings