J. Differential Equations 253 (2012) 1489–1543

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

# Lower dimensional invariant tori for quasiperiodically forced circle diffeomorphisms ☆

Jing Wang

*Department of Mathematics, Nanjing University, Nanjing 210093, China*

ARTICLE INFO

*Article history:*
Received 31 December 2011
Revised 10 May 2012
Available online 17 May 2012

ABSTRACT

For any analytic quasiperiodically forced circle diffeomorphisms $(\omega, \langle \frac{p}{q}, \omega\rangle + \varepsilon f)$, where $f$ is fixed and $\varepsilon$ is small, we show that if $\omega$ is Diophantine and the fibred rotation number of the diffeomorphism remains constant in a unilateral neighborhood of $\varepsilon = 0$ (i.e., there is a unilateral phase-locking at $\varepsilon = 0$), then the diffeomorphism has at least one analytic $q$-invariant torus, provided $\varepsilon$ is small enough.



## 1. Introduction

We study the quasiperiodically forced circle homeomorphisms, which are isotopic to the identity of the form

$$\begin{aligned}(\omega, f): \mathbb{T}^m \times \mathbb{T} &\to \mathbb{T}^m \times \mathbb{T},\\ (\varphi, \theta) &\mapsto \big(\varphi + \omega, \theta + f(\varphi, \theta)\big), \end{aligned} \tag{1}$$

where $\mathbb{T}^m = \mathbb{R}^m / 2\pi\mathbb{Z}^m$, the fibre maps $f_\varphi(\cdot) := \cdot + f(\varphi, \cdot)$ are all orientation preserving circle homeomorphisms and $\omega = (\omega_1, \ldots, \omega_m)$ with $(1, \omega_1, \ldots, \omega_m)$ being rational independent. We denote the set of these homeomorphisms by $\mathcal{F}$. If $f$ is analytic, we call $(\omega, f)$ a $C^\omega$-*qpf* circle diffeomorphism, denoted by $\mathcal{F}^\omega$ the set of these diffeomorphisms.

Skew-products transformations like these occur in various situations in physics and have been extensively studied as a source of examples of interesting dynamics. For instance, the Harper map,

☆ The work was supported by NNSF of China (Grant 10531050), NNSF of China (Grant 11031003) and a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

*E-mail address:* jingwang018@gmail.com.

0022-0396/$ – see front matter 
http://dx.doi.org/10.1016/j.jde.2012.05.005

which is related to some discrete Schrödinger operators with quasiperiodic potential, also serves as a model of quasiperiodic crystals [8]. The *qpf* Arnold circle map is another example that can be taken as a model of oscillators forced with two or more incommensurate frequencies [5].

In the one-dimensional case (i.e., $m = 0$), namely, the unforced circle map, there are fruitful results, one of which is an elegant classification theorem, the Poincaré classification theorem. It asserts that the circle map $f$ has a periodic orbit of period $q$ if and only if the rotation number is rational $\rho = \frac{p}{q}$ [1].

It is natural to generalize this result to higher dimensions, that is to homeomorphisms of the torus $\mathbb{T}^n$. However, even when $n = 2$, the problem is much more complicated, since there is no rotation number. The corresponding concept is the rotation vector, yet in most cases, the rotation vectors form a rotation set rather than a single vector, since they depend on the orbit. Thus, in this paper, we focus on the quasiperiodically forced circle diffeomorphisms, for which the fibred rotation number is well defined [10].

In the topological aspect, due to the aperiodicity of the forcing rotation, there is no fixed or periodic points for homeomorphisms (1). The appropriate analogue concept is the *invariant graph* or *p, q-invariant graph* [12]. However, an *invariant graph* does not necessarily have any topological structure. So, Jäger and Stark [12] generalized this concept to *invariant strip* and for $m = 1$, they gave a Poincaré-like classification of the *qpf* circle homeomorphisms:

- If $(\omega, f - id)$ is *ρ-bounded*, then either there exists a *p, q-invariant strip* and $(\omega, \rho_f(\omega, f - id))$ is rationally dependent or $(\omega, f - id)$ is *semi-conjugate* to the irrational torus translation $(\omega, \rho_f(\omega, f - id))$.
- If $(\omega, f - id)$ is *ρ-unbounded*, then neither of these alternatives can occur and the map is always topologically transitive.

Recently, Huang and Yi [11] gave an elaborate discussion of the dynamical and topological behaviors of minimal sets in almost periodically forced circle flow, which is more general than quasiperiodically forced circle flow.

However, there is few other information for *qpf* circle homeomorphisms, such as the regularity about the invariant bounding graphs of the invariant strip, even in the analytic category. Numerical experiments show that it is possible to have $\rho$-bounded *qpf*-circle homeomorphism with rational rotation numbers, but no continuous invariant graph [7]. Yet, we know little about the regularity of the invariant bounding graph in stronger conditions, for instance $(\omega, f) \in \mathcal{F}^\omega$.

In the analytic aspect, Herman [9] proved that if the base frequency $\omega$ is Diophantine and the fibred rotation number $\rho$ is Diophantine with respect to $\omega$, then the system $(\omega, \rho + f)$ is linearizable, provided $f$ is analytically small. Recently, Krikorian et al. [13] generalized this result, proving that if the base frequency $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is not super-Liouvillean and the fibred rotation number $\rho$ is Diophantine with respect to $\alpha$, then the system $(\alpha, \rho + f)$ is $C^\infty$ linearizable for $f$ analytically small. Notice that, all of the above results requires that the fibred rotation number is Diophantine with respect to the base frequency.

So far, little is known when the fibred rotation number is rational with respect to the base frequency in the analytic case. If the diffeomorphism comes from the projective action of the analytic quasiperiodic $SL(2, \mathbb{R})$ cocycle, it does have an invariant torus in the local region, since Eliasson [6] proved that if the $SL(2, \mathbb{R})$ cocycle is close to constant, the base frequency $\omega$ is Diophantine, the fibred rotation number is Diophantine or rational with respect to $\omega$, then it is reducible. This result motivated us to consider the existence of invariant torus for general $C^\omega$-*qpf* circle diffeomorphisms, when $(\omega, \rho_f)$ is rational dependent. In this paper, we are concerned with the $C^\omega$-*qpf* circle diffeomorphisms $(\omega, \langle \frac{p}{q}, \omega \rangle + \varepsilon f)$ which is close to constant one $(\omega, \langle \frac{p}{q}, \omega \rangle)$. We prove that if $\omega$ is Diophantine and the fibred rotation number remains constant in a neighborhood of $\varepsilon = 0$, then for sufficiently small $\varepsilon$, the map has at least one invariant torus of the form $\theta = u(\varphi)$.