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Sturm–Liouville boundary value problems with operator potentials and unitary equivalence

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A B S T R A C T

Consider the minimal Sturm–Liouville operator $A = A_{\min}$ generated by the differential expression

$$\mathcal{A} := -\frac{d^2}{dt^2} + T$$

in the Hilbert space $L^2(\mathbb{R}_+, \mathcal{H})$ where $T = T^* \geqslant 0$ in $\mathcal{H}$. We investigate the absolutely continuous parts of different self-adjoint realizations of $\mathcal{A}$. In particular, we show that Dirichlet and Neumann realizations, A^D and A^N, are absolutely continuous and unitary equivalent to each other and to the absolutely continuous part of the Krein realization. Moreover, if $\inf \sigma_{\text{ess}}(T) = \inf \sigma(T)$, then the part $\widetilde{A}^{ac} E_{\widetilde{A}}(\sigma(A^D))$ of any self-adjoint realization $\widetilde{A}$ of $\mathcal{A}$ is unitarily equivalent to A^D. In addition, we prove that the absolutely continuous part $\widetilde{A}^{ac}$ of any realization $\widetilde{A}$ is unitarily equivalent to A^D provided that the resolvent difference $(\widetilde{A} - i)^{-1} - (A^D - i)^{-1}$ is compact. The abstract results are applied to elliptic differential expressions in the half-space.

Contents

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1. Introduction

Let T be a non-negative unbounded self-adjoint operator in an *infinite* dimensional separable Hilbert space $\mathcal{H}$. We consider the minimal Sturm–Liouville operator A generated by the differential expression

$$\mathcal{A} := -\frac{d^2}{dt^2} + T \tag{1.1}$$

in the Hilbert space $\mathfrak{H} := L^2(\mathbb{R}_+, \mathcal{H})$ of $\mathcal{H}$-valued square summable vector-valued functions. Following [19,20] the minimal operator $A := A_{\min}$ is introduced to be the closure of the operator A' defined by

$$A' := \mathcal{A} \restriction \mathcal{D}_0, \quad \mathcal{D}_0 := \left\{ \sum_{1 \leqslant j \leqslant k} \phi_j(t) h_j \colon \phi_j \in W_0^{2,2}(\mathbb{R}_+),\ h_j \in \mathrm{dom}(T),\ k \in \mathbb{N} \right\}, \tag{1.2}$$

where $W_0^{2,2}(\mathbb{R}_+) := \{\phi \in W^{2,2}(\mathbb{R}_+)\colon\ \phi(0) = \phi'(0) = 0\}$, that is, $A_{\min} := \overline{A'}$. It is easily seen that A is a closed non-negative symmetric operator in $\mathcal{H}$ with equal deficiency indices $n_\pm(A) = \dim(\mathcal{H})$. The adjoint operator A^* of $A = A_{\min}$ is the maximal operator denoted by $A_{\max}$. Self-adjoint extensions of A (are also called self-adjoint realizations of $\mathcal{A}$) were investigated for the first time by M.L. Gorbachuk [19] in the case of finite intervals I. He proved that the traces of vector-functions $f \in \mathrm{dom}(A_{\max})$ belong to the space $\mathcal{H}_{-1/4}(T)$, cf. (5.2) and, in particular, $\mathrm{dom}(A_{\max})$ is not contained in the Sobolev space $W^{2,2}(I, \mathcal{H})$. Based on this result he constructed a boundary triplet for the operator $A_{\max} = A_{\min}^* = A^*$ in the Hilbert space $L^2(I, \mathcal{H})$ and described all self-adjoint realizations of $\mathcal{A}$ in terms of boundary conditions. These results are similar to those for elliptic operators in domains with smooth boundaries, cf. [4,24,34], and go back to classical papers of M.I. Višik [42] and G. Grubb [23].