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Monotone traveling wavefronts of the KPP-Fisher delayed equation

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A B S T R A C T

In the early 2000's, Gourley (2000), Wu et al. (2001), Ashwin et al. (2002) initiated the study of the positive wavefronts in the delayed Kolmogorov–Petrovskii–Piskunov–Fisher equation

$$u_t(t,x) = \Delta u(t,x) + u(t,x)\big(1 - u(t-h,x)\big),$$
$$u \geqslant 0,\ x \in \mathbb{R}^m. \qquad (*)$$

Since then, this model has become one of the most popular objects in the studies of traveling waves for the monostable delayed reaction–diffusion equations. In this paper, we give a complete solution to the problem of existence and uniqueness of monotone waves in Eq. $(*)$. We show that each monotone traveling wave can be found via an iteration procedure. The proposed approach is based on the use of special monotone integral operators (which are different from the usual Wu–Zou operator) and appropriate upper and lower solutions associated to them. The analysis of the asymptotic expansions of the eventual traveling fronts at infinity is another key ingredient of our approach.

1. Introduction and main results

It is well known that the traveling waves theory was initiated in 1937 by Kolmogorov, Petrovskii, Piskunov [22] and Fisher [15] who studied the wavefront solutions of the diffusive logistic equation

$$u_t(t,x) = \Delta u(t,x) + u(t,x)\big(1 - u(t,x)\big), \quad u \geqslant 0,\ x \in \mathbb{R}^m. \tag{1}$$

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We recall that the classical solution $u(x,t) = \phi(\nu \cdot x + ct)$, $\|\nu\| = 1$, is a wavefront (or a traveling front) for (1), if the profile function ϕ is positive and satisfies $\phi(-\infty) = 0$, $\phi(+\infty) = 1$.

The existence of the wavefronts in (1) is equivalent to the presence of positive heteroclinic connections in an associated second order non-linear differential equation. The phase plane analysis is the natural geometric way to study these heteroclinics. The method is conclusive enough to demonstrate that (a) for every $c \geqslant 2$, the KPP-Fisher equation has exactly one traveling front $u(x,t) = \phi(\nu \cdot x + ct)$; (b) Eq. (1) does not have any traveling front propagating at the velocity $c < 2$; (c) the profile ϕ is necessarily strictly increasing function.

The stability of traveling fronts in (1) represents another important aspect of the topic: however, we do not discuss it here. Further reading and relevant information can be found in [6,24,30,39].

Eq. (1) can be viewed as a natural extension of the ordinary logistic equation $u'(t) = u(t)(1 - u(t))$. An important improvement of this growth model was proposed by Hutchinson [20] in 1948 who incorporated the maturation delay $h > 0$ in the following way:

$$u'(t) = u(t)\big(1 - u(t-h)\big), \quad u \geqslant 0. \tag{2}$$

This model is now commonly known as the Hutchinson' s equation. Since then, the delayed KPP-Fisher equation or the diffusive Hutchinson's equation

$$u_t(t,x) = \Delta u(t,x) + u(t,x)\big(1 - u(t-h,x)\big), \quad u \geqslant 0,\ x \in \mathbb{R}^m, \tag{3}$$

is considered as a natural prototype of delayed reaction–diffusion equations. It has attracted the attention of many authors, see [2,4,12,16,17,19,25,35,38,40]. In particular, the existence of traveling fronts connecting the trivial and positive steady states in (3) (and its non-local generalizations) was studied in [2,4,8,12,18,29,35,38]. Observe that the biological meaning of u is the size of an adult population, therefore *only* non-negative solutions of (3) are of interest. It is worth to mention that there is another delayed version of Eq. (1) derived by Kobayashi [21] from a branching process:

$$u_t(t,x) = \Delta u(t,x) + u(t-h,x)\big(1 - u(t,x)\big), \quad u \geqslant 0,\ x \in \mathbb{R}^m.$$

However, since the right-hand side of this equation is monotone increasing with respect to the delayed term, the theory of this equation is fairly different (and seems to be simpler) from the theory of (3), see [32,38,41].

This paper deals with the problem of existence and uniqueness of monotone wavefronts for Eq. (3). The phase plane analysis does not work now because of the infinite dimension of phase spaces associated to delay equations. Recently, the existence problem was considered by using two different approaches. The first method, which was proposed in [38], uses the positivity and monotonicity properties of the integral operator

$$(A\phi)(t) = \frac{1}{\epsilon'}\left\{ \int_{-\infty}^{t} e^{r_1(t-s)}(\mathcal{H}\phi)(s)\,ds + \int_{t}^{+\infty} e^{r_2(t-s)}(\mathcal{H}\phi)(s)\,ds \right\}, \tag{4}$$

where $(\mathcal{H}\phi)(s) = \phi(s)(\beta + 1 - \phi(s-h))$ for some appropriate $\beta > 1$, and $\epsilon' = \epsilon(r_2 - r_1)$ with $r_1 < 0 < r_2$ satisfying $\epsilon z^2 - z - \beta = 0$, and $\epsilon^{-1/2} = c > 0$ is the front velocity. A direct verification shows that the profiles $\phi \in C(\mathbb{R}, \mathbb{R}_+)$ of traveling waves are completely determined by the integral equation $A\phi = \phi$. Wu and Zou have found a subtle combination of the usual and the Smith and Thieme nonstandard orderings on an appropriate profile set $\Gamma^* \subset C(\mathbb{R}, (0,1))$ which allowed them (under specific quasimonotonicity conditions) to indicate a pair of upper and lower solutions $\phi^{\pm}$ such that $\phi^- \leqslant A^{j+1}\phi^+ \leqslant A^j\phi^+$, $j = 0, 1, \ldots$. Then the required traveling front profile is given by $\phi = \lim A^j\phi^+$. More precisely, in [38, Theorem 5.1.5], Wu and Zou established the following