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Sharp existence results for mean field equations with singular data

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ABSTRACT

Let Ω be a simply connected, open and bounded domain in $\mathbb{R}^2$. We are concerned with the nonlinear elliptic problem

$$\begin{cases} -\Delta v = 8\pi \dfrac{e^v}{\int_\Omega e^v} - 4\pi \displaystyle\sum_{j=1}^m \alpha_j \delta_{p_j} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $\alpha_j > 0$, δ_{p_j} denotes the Dirac mass with singular point p_j and $\{p_1, \dots, p_m\} \subset \Omega$. We provide necessary and sufficient conditions for the existence of solutions to (0.1). Our result is the two dimensional version of the sharp existence/nonexistence result obtained in Druet (2002) [13] for elliptic equations with critical exponent in dimension 3. In particular, we prove that the set $\Omega_+^m(\underline{\alpha})$ is open, where, for a given $\underline{\alpha} = (\alpha_1, \dots, \alpha_m) \subset (0, +\infty) \times \dots \times (0, +\infty)$, $\Omega_+^m(\underline{\alpha}) = \{(p_1, \dots, p_m) \mid \text{ problem (0.1) has a solution}\} \subset \Omega \times \dots \times \Omega$.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a simply connected, open and bounded domain and $\{p_1, \dots, p_m\} \subset \Omega$ be any finite subset. We are concerned with the existence of solutions for

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$$\begin{cases} -\Delta v = \lambda \dfrac{e^v}{\int_\Omega e^v} - 4\pi \displaystyle\sum_{j=1}^m \alpha_j \delta_{p_j} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in case $\lambda = 8\pi$ and $\alpha_j \in (0, +\infty)$, $\forall j \in \{1, \dots, m\}$.

The analysis of (1.1) has recently attracted a lot of attention due its many applications in mathematical physics. We refer the reader to [2,4,5,7–12,14,15,17,19–22], and the references quoted therein for further details. In particular we refer to [7,8,16] and the introduction of [4] for the application of (1.1) to the analysis of vortex-type configurations in turbulent Euler flows.

We will not discuss here issues related with non-smooth domains. Therefore, unless otherwise specified we will assume that Ω is of class C^2.

We will denote by $z = x_1 + ix_2$ and $p_j = p_{j,1} + ip_{j,2}$ the complex coordinates corresponding to $(x_1, x_2) \in \Omega$, $(p_{j,1}, p_{j,2}) \in \Omega$, and by $D = \{z \in \mathbb{R}^2 \mid |z| < 1\}$ the open unit disk.

We will often need to use conformal mappings from D to Ω. To avoid any possible ambiguity, in case Ω itself is the unit disk, we will denote it by B_1. In particular, for any fixed $p \in \Omega$, we will denote by $f_p : D \mapsto \Omega$, any Riemann map which satisfies $f_p(0) = p$, and set $g_p = f_p^{-1} : \Omega \mapsto D$ to be its inverse.

For any fixed $\alpha > 0$, $p \in \Omega$ and any $r > 0$ small enough, let $G_\Omega(z, p) \in C^0(\overline{\Omega} \setminus B_r(p))$ be the unique solution of

$$\begin{cases} -\Delta G_\Omega(z, p) = \delta_p & \text{in } \Omega, \\ G_\Omega(z, p) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The continuity assumption ensures that G_Ω is uniquely defined by $G_\Omega(z, p) = -\frac{1}{2\pi} \log |g_p(z)|$.

Let $\tilde{G}_\Omega$ denote the regular part of $G_\Omega(z, p)$, and set $z = f_p(w)$. Then, we have

$$\tilde{G}_\Omega(z, p) = G_\Omega(z, p) + \frac{1}{2\pi} \log |z - p| = -\frac{1}{2\pi} \log \frac{|g_p(z)|}{|z - p|} = \frac{1}{2\pi} \log \frac{|f_p(w) - f_p(0)|}{|w|},$$

and we define

$$\gamma_\Omega(z) = \tilde{G}_\Omega(z, z) = \frac{1}{2\pi} \log \frac{(1 - |g_p(z)|^2)}{|g_p'(z)|} = \frac{1}{2\pi} \log\bigl(1 - |w|^2\bigr)\bigl|f_p'(w)\bigr|$$

to be the corresponding Robin function.

We define v to be a solution for (1.1) if $u := v + 4\pi \sum_{j=1,\dots,m} \alpha_j G_\Omega(z, p_j)$ is an $H_0^1(\Omega)$, weak solution for

$$\begin{cases} -\Delta u = \lambda \dfrac{V e^u}{\int_\Omega V e^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where

$$V(z) = \exp\Biggl(-4\pi \sum_{j=1,\dots,m} \alpha_j G_\Omega(z, p_j)\Biggr). \tag{1.4}$$

Two crucial results (Theorems 1.1 and 1.4 below) will be used whose proofs will not be discussed here. We refer the reader to the remarks following the corresponding statements for further details.