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# Formation of delta shocks and vacuum states in the vanishing pressure limit of Riemann solutions to the perturbed Aw–Rascle model ☆

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ABSTRACT

A traffic flow model describing the formation and dynamics of traffic jams was introduced by Berthelin et al., which consists of a constrained pressureless gas dynamics system and can be derived from the Aw–Rascle model under the constraint condition $\rho \leqslant \rho^*$ by letting the traffic pressure vanish. In this paper, we give up this constraint condition and consider the following form

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \big(\rho u + \varepsilon p(\rho)\big)_t + \big(\rho u^2 + \varepsilon u p(\rho)\big)_x = 0, \end{cases}$$

in which $p(\rho) = \rho^\gamma$ with $\gamma > 1$.

The formal limit of the above system is the pressureless gas dynamics system in which the density develops delta-measure concentration in the Riemann solution. However, the propagation speed and the strength of the delta shock wave in the limit situation are different from the classical results of the pressureless gas dynamics system with the same Riemann initial data.

In order to solve it, the perturbed Aw–Rascle model is proposed as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \left(\rho u + \dfrac{\varepsilon}{\gamma} p(\rho)\right)_t + \big(\rho u^2 + \varepsilon u p(\rho)\big)_x = 0, \end{cases}$$

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whose behavior is different from that of the Aw–Rascle model. It is proved that the limits of the Riemann solutions of the perturbed Aw–Rascle model are exactly those of the pressureless gas dynamics model.



## 1. Introduction

### 1.1. The Aw–Rascle model and its singular limit

The Aw–Rascle (AR) model in the conservative form [3] can be expressed as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \big(\rho\big(u + p(\rho)\big)\big)_t + \big(\rho u\big(u + p(\rho)\big)\big)_x = 0, \end{cases} \tag{1.1}$$

where $\rho, u$ represent the density and the velocity, respectively; the velocity offset $p$ takes the form $p(\rho) = \rho^\gamma$ with $\gamma > 0$. The AR model describes a single lane traffic model and therefore the velocity is assumed to be a bounded nonnegative function.

The AR model (1.1) is now widely used to study the formation and dynamics of traffic jams. It was proposed to remedy the deficiencies of second order models of car traffic pointed out by Daganzo [14] and has been independently derived by Zhang [49]. The derivation of the model from a microscopic Follow-the-Leader (FL) model through a scaling limit was given in [2]. The AR model resolves all the obvious inconsistencies and explains instabilities in car traffic flow, especially near the vacuum, i.e., for light traffic with few slow drivers. It is also the basis for the multi-lane traffic flow model [20,21], the model for a road network with unidirectional flow [19,24] and the hybrid traffic flow model [36].

Recently, the singular limit behavior has been investigated for the AR model (1.1) by changing $p$ into $\varepsilon p$ and taking $p(\rho) = (\frac{1}{\rho} - \frac{1}{\rho^*})^{-\gamma}$ with the density constraint $\rho \leqslant \rho^*$, where the maximal density $\rho^*$ corresponds to a bumper. In order to describe the formation and dynamics of traffic jams, a constrained pressureless gas dynamics (CPGD) model was proposed by Berthelin et al. [4] as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (u + \overline{p})_t + u(u + \overline{p})_x = 0, \end{cases} \tag{1.2}$$

with the condition

$$0 \leqslant \rho \leqslant \rho^*, \qquad \overline{p} \geqslant 0, \qquad \big(\rho^* - \rho\big)\overline{p} = 0. \tag{1.3}$$

In the above, $\rho$ denotes the density of vehicles and it is assumed that the maximal density constraint $\rho^*$ is independent of the velocity $u$. The quantity $\overline{p}$ can be regarded as the Lagrange multiplier of the constraint $\rho \leqslant \rho^*$ and is nonzero only when $\rho = \rho^*$, namely it arrives at the maximal density constraint which is in congested situations and cars are then forced to spread into clusters.

The CPGD model can be derived from the so-called rescaled modified Aw–Rascle (RMAR) model

$$\begin{cases} \rho^\varepsilon_t + \big(\rho^\varepsilon u^\varepsilon\big)_x = 0, \\ \big(u^\varepsilon + \varepsilon p\big(\rho^\varepsilon\big)\big)_t + u^\varepsilon\big(u^\varepsilon + \varepsilon p\big(\rho^\varepsilon\big)\big)_x = 0, \end{cases} \tag{1.4}$$

where $p(\rho^\varepsilon)$ takes the form

$$p\big(\rho^\varepsilon\big) = \left(\frac{1}{\rho^\varepsilon} - \frac{1}{\rho^*}\right)^{-\gamma}, \quad \rho^\varepsilon \leqslant \rho^*. \tag{1.5}$$