J. Differential Equations 252 (2012) 1392–1402

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Radial symmetry of positive solutions for semilinear elliptic equations in the unit ball via elliptic and hyperbolic geometry

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ARTICLE INFO

Article history:
Received 30 January 2011
Revised 29 September 2011
Available online 10 October 2011

MSC:
35B06
35B09
35J15

Keywords:
Elliptic equations
Positive solutions
Radial symmetry
Elliptic and hyperbolic geometry

ABSTRACT

Let $n \in \mathbb{N}$ with $n \geqslant 2$, $a \in (-1,0) \cup (0,1]$ and $f : (0,1) \times (0,\infty) \to \mathbb{R}$ such that for each $u \in (0,\infty)$, $r \mapsto (1+ar^2)^{(n+2)/2} f(r,(1+ar^2)^{-(n-2)/2}u) : (0,1) \to \mathbb{R}$ is nonincreasing. We show that each positive solution of

$$\Delta u + f\big(|x|, u\big) = 0 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B$$

is radially symmetric, where B is the open unit ball in $\mathbb{R}^N$.

1. Introduction

We consider symmetry and monotonicity properties of positive solutions of the problem

$$\begin{cases} \Delta u + f\big(|x|, u\big) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

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1 This work is partially supported by the Grant-in-Aid for Scientific Research (C) (No. 21540214) from Japan Society for the Promotion of Science.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.10.001

where $f : (0,1) \times (0,\infty) \to \mathbb{R}$ and B is the open unit ball in $\mathbb{R}^n$, i.e., $B = \{x \in \mathbb{R}^n\colon\ |x| < 1\}$. Celebrated Gidas–Ni–Nirenberg's theorem [10] shows that if for each $u \in (0,\infty)$, $r \mapsto f(r,u) : (0,1) \to \mathbb{R}$ is nonincreasing, then any $C^2(\overline{B})$-positive solution of (1.1) is radially symmetric. Many researchers studied such symmetry properties; see [1–5,7–11,13–24] and others. In some of them, geometry plays an important role. In [19], Naito, Nishimoto and Suzuki considered the case that $n = 2$ (i.e., B is the open unit ball in $\mathbb{R}^2$) and $(1-r^2)^2 f(r,u) : (0,1) \to \mathbb{R}$ is decreasing for each $u \in (0,\infty)$. Using hyperbolic geometry, they showed each positive solution of (1.1) is radially symmetric. Naito and Suzuki [20] extended their result to the case that $n \geqslant 2$ and $r \mapsto (1-r^2)^{(n+2)/2} f(r,(1-r^2)^{-(n-2)/2}u)$ is decreasing for each $r \in (0,1)$ and $u \in (0,\infty)$. Almeida, Ge and Orlandi [1] gave a similar result. (Although the arguments in [1] seem to be fine, the assumption (1.2) in [1, Theorem 1.1] is not correct.)

In this paper, we consider not only hyperbolic geometry but also elliptic geometry, and we show a symmetric result of (1.1). Since we want to treat a wide class of solutions of (1.1), we recall the definition of a strong solution in [12, p. 219]. We say a function $u \in L^1_{\mathrm{loc}}(B)$ is said to be a strong solution of (1.1) if each first and second derivative of u in the sense of distribution belongs to $L^1_{\mathrm{loc}}(B)$ and u satisfies (1.1) almost everywhere in B. Now, we show our result.

Theorem 1. *Let $n \in \mathbb{N}$ with $n \geqslant 2$, $a \in (-1,0) \cup (0,1]$ and $f : (0,1) \times (0,\infty) \to \mathbb{R}$ such that*

(i) *for each $u \in (0,\infty)$, $r \mapsto (1+ar^2)^{(n+2)/2} f(r,(1+ar^2)^{-(n-2)/2}u)$ is nonincreasing,*
(ii) *for each $r_0 \in (0,1)$ and $M \in (0,\infty)$,*

$$\sup\left\{\left|\frac{f(r,u_1) - f(r,u_2)}{u_1 - u_2}\right|\colon\ (r,u_1,u_2) \in (r_0,1) \times (0,M]^2,\ u_1 \neq u_2\right\} < \infty.$$

Let $u \in W^{2,n}_{\mathrm{loc}}(B) \cap C(\overline{B})$ be a positive strong solution of (1.1). Then u is radially symmetric. Moreover, if $u \in C^1(B)$ then $((1+ar^2)^{(n-2)/2}u)_r < 0$ for $r = |x| \in (0,1)$.

Remark 1. For related results, we give some comments.

(i) The case $a = 0$ is nothing but the Gidas–Ni–Nirenberg's theorem in [10] for the case of B. The case $a = -1$ is studied in [19,20] under the assumption that for each $u \in (0,\infty)$, $r \mapsto (1-r^2)^{(n+2)/2} f(r,(1-r^2)^{-(n-2)/2}u) : (0,1) \to \mathbb{R}$ is decreasing instead of nonincreasing. The reason why nonincreasingness is not enough is that (2.5) does not hold in the case $a = -1$; see the proofs of Lemmas 2 and 3 below. We also note that two coefficient functions in (3.1) are not essentially bounded in the case of $a = -1$ and hence an additional device is needed to derive the symmetry result. For the details, see [20].
(ii) As we stated, the assumption (1.2) in [1, Theorem 1.1] is not correct. However, since the domain in [1, Theorem 1.1] is an open ball whose radius is less than 1, from the arguments in [1], we can see that the result corresponds to the case $a \in (-1,0)$ of our result.
(iii) We can apply our result to the equations

$$\begin{cases} \Delta_g u + f\big(d(x,x_0),u\big) = 0 & \text{in } D,\\ u = 0 & \text{on } \partial D,\end{cases}$$

where we consider hyperbolic space $\mathbb{H}^n$ or sphere $\mathbb{S}^n$, D is a geodesic ball in $\mathbb{H}^n$ or $\mathbb{S}^n$ whose center is x_0 and the radius is r_0, Δ_g is the corresponding Laplace–Beltrami operator, $f : (0,r_0) \times \mathbb{R} \to \mathbb{R}$ and d is the Riemannian distance. In order to find how to apply our result, see the proof of [8, Theorem 5], in which Gidas–Ni–Nirenberg's theorem is applied. For related results, see also [1,13,23].