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Dispersive estimate for the 1D Schrödinger equation with a steplike potential

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ABSTRACT

We prove a sharp dispersive estimate

$$\left|P_{ac}u(t,x)\right| \leqslant C|t|^{-1/2}\cdot \left\|u(0)\right\|_{L^1(\mathbb{R})}$$

for the one dimensional Schrödinger equation

$$iu_t - u_{xx} + V(x)u + V_0(x)u = 0,$$

where $(1+x^2)V \in L^1(\mathbb{R})$ and V_0 is a step function, real valued and constant on the positive and negative real axes.

1. Introduction

The *dispersive estimate* for the Schrödinger equation on $\mathbb{R}^n$, $n \geqslant 1$,

$$iu_t - \Delta u = 0, \qquad u(0,x) = f(x)$$

states that, for all $f \in L^1(\mathbb{R}^n)$ and $t \neq 0$, the solution satisfies

$$\left|u(t,x)\right| \leqslant C|t|^{-\frac{n}{2}}\|f\|_{L^1(\mathbb{R}^n)}. \tag{1.1}$$

The sharp constant is $C = (4\pi)^{-n/2}$. Estimate (1.1) is elementary and follows from the explicit form of the fundamental solution; nevertheless, it represents the starting point for a large number of

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important developments including Strichartz estimates and the local and global well posedness theory for nonlinear Schrödinger equations. Thus the problem of extending (1.1) to more general equations has received a great deal of attention.

Potential perturbations of the form

$$iu_t - \Delta u + V(x)u = 0$$

(on $\mathbb{R}^n$, $n \geqslant 3$) were considered in many papers, starting with [21], with improvements at several reprises (see e.g. [30–32,20,12], see also [11] for first order perturbations). Focusing on the one dimensional case

$$iu_t - u_{xx} + V(x)u = 0, \tag{1.2}$$

which is the subject of this paper, the dispersive estimate was first proved by Weder [27]. The more general fact that the wave operator associated to $-\frac{d^2}{dx^2} + V(x)$ is bounded on $W^{k,p}$ for all $1 < p < \infty$ was proved in [26] (see also [2]). Optimal conditions on the potential implying the dispersive estimate (1.1) were obtained by Goldberg and Schlag [20] who considered potentials satisfying $(1 + x^2)V \in L^1(\mathbb{R})$, or the weaker condition $(1 + |x|)V \in L^1(\mathbb{R})$ plus an additional nonresonant condition at 0. Under the same assumptions on V, the L^p boundedness of the wave operator was proved by D'Ancona and Fanelli [10]. We also mention that potentials with slower decay present new phenomena as evidenced in [5,6].

Thus the problem of dispersive estimates for potential perturbations is essentially settled in 1D if the potential is small in the appropriate sense at infinity. Notice that we can add a real constant to V without modifying the dispersive properties via the gauge transformation $u \to ue^{i\lambda t}$. Hence a more precise statement is that dispersion has been proved for potentials having the same asymptotic behaviour at $x \to \pm\infty$.

Here we consider a more general kind of potential with possibly different asymptotic behaviours at $+\infty$ and $-\infty$. We call a *steplike potential* a potential of the form

$$V(x) + V_0(x)$$

where

$$V \in L^1(\mathbb{R}), \qquad V_0(x) = \begin{cases} V_-: & x < 0, \\ V_+: & x > 0, \end{cases} \quad V_-, V_+ \in \mathbb{R}.$$

It is not restrictive to assume $V_- < V_+$ as we shall do from now on.

In the physical literature, steplike potentials are also called *barrier potentials* and are used to model the interaction of particles with the boundary of solids (see [19] for a general discussion of problems with nontrivial asymptotics).

Steplike potentials occur also in general relativity. An example is given by the radial Klein–Gordon equation $\Box_g u - m^2 u = 0$ on a (radial) Schwarzschild background

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2.$$

Here $M > 0$ is related to the mass of the black hole and $r > 2M$ is a radial variable. If we introduce the Regge–Wheeler coordinate

$$s = r + 2M \log\left(\frac{r - 2M}{2M}\right) \tag{1.3}$$