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Blow-up analysis of a Finsler–Liouville equation in two dimensions ☆

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ABSTRACT

In this paper, we establish a blow-up analysis for the Finsler–Liouville equation

$$-Qu = V(x)e^u$$

in two dimensions. Here the operator Q is the so-called *Finsler–Laplacian*, which is a generalized Laplacian. As a byproduct, we obtain an existence result.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$. The Moser–Trudinger inequality says that the functional

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - 8\pi \log \int_\Omega e^u \tag{1}$$

is bounded below for any $u \in W_0^{1,2}(\Omega)$. The corresponding Euler–Lagrange equation for $J(u)$ is the so-called Liouville equation

$$-\Delta u = \lambda \frac{e^u}{\int_\Omega e^u} \tag{2}$$

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for some constant $\lambda > 0$, which was first studied by Liouville in 1853 in [31]. The functional (1) and Eq. (2) have been intensively studied by many mathematicians, for there are many applications in geometric and physical problems, for example, in the problem of prescribing Gaussian curvature [9,12,8], in the theory of the mean field equation [16,17,10,11,19,32] and in the Chern–Simons theory [38,41,18,39,34]. See also survey articles [30] and [42].

In the celebrated paper by Brezis and Merle [6], they initiated the study of the blow-up analysis for the Liouville equation

$$-\Delta u = V(x)e^u \tag{3}$$

with $V(x) \in L^p(\Omega)$ and $e^u \in L^{p'}$ for $1 < p \leqslant \infty$ and $p' = \frac{p}{p-1}$. They first showed that any solution of (3) belongs to L^∞, and further they analyzed the convergence of a sequence of solutions of (3) and obtained a compactness-concentration type result. Their results initiate many works on the asymptotic behavior of blow-up solutions and also the existence of solutions of Liouville equation (2).

The aim of the present paper is to generalize the blow-up analysis for Eq. (2) to a Liouville type equation with *Finsler–Laplacian* (or *anisotropic Laplacian*). In other words, we consider the following quasilinear equations,

$$-Qu = V(x)e^u, \tag{4}$$

and

$$-Qu = \lambda \frac{e^u}{\int_\Omega e^u}. \tag{5}$$

The operator Q is the so-called *Finsler–Laplacian*, which is defined as follows. If we consider a domain in $\mathbb{R}^n$, the operator Qu is defined by

$$Qu := \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(F(\nabla u)F_{\xi_i}(\nabla u)\big),$$

where $F \in C^2(\mathbb{R}^n \setminus \{0\})$ is a convex and homogeneous function and $F_{\xi_i} = \frac{\partial F}{\partial \xi_i}$. In the isotropic case, i.e., $F(\xi) = |\xi|$, Q is nothing but the ordinary Laplacian. This operator is closely related to a smooth, convex hypersurface in $\mathbb{R}^n$, which is called the *Wulff shape* (or *equilibrium crystal shape*) of F. The study of the Wulff shape was initiated in Wulff's work [46] on crystal shapes and minimization of a surface energy $\int_{\partial\Omega} F(\nu)\,d\sigma$ among regular domains Ω. Here, ν is the outer normal vector field of $\partial\Omega$. The Wulff shape $\mathcal{W}_F$ is the unique minimum (up to translations) of the surface energy (see e.g. [6,23,43]). For exact assumptions on F and the definition of Wulff shape, see Section 2. This operator Q was studied by many mathematicians [1–3,5,21,44,45]. We call Eqs. (4) and (5) *Finsler–Liouville equations*.

As in the isotropic case, Eq. (4) has a corresponding functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega F(\nabla u)^2 - \lambda \log \int_\Omega e^u,$$

for $u \in W_0^{1,2}(\Omega)$. By using a convex symmetrization approach proposed in [2] and an argument of Moser [33], we first prove a Moser–Trudinger type inequality, Theorem 3.1. As a direct consequence, we have that $J_\lambda(u)$ is bounded below if and only if $\lambda \leqslant 8\kappa$. Here κ is the area of a Wulff ball of radius 1 (see Section 2). In the isotropic case, $\kappa = \pi$.

To study the asymptotic behavior of convergence and the existence of solutions, we first prove the following Brezis–Merle type compactness-concentration result.