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Blow-up for a semilinear parabolic equation with large diffusion on $\mathbf{R}^N$. II

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ABSTRACT

We are concerned with the Cauchy problem for a semilinear heat equation,

$$\begin{cases} \partial_t u = D\Delta u + |u|^{p-1}u, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = \lambda + \varphi(x), & x \in \mathbf{R}^N, \end{cases} \tag{P}$$

where $D > 0$, $p > 1$, $N \geqslant 3$, $\lambda > 0$, and $\varphi \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N, (1+|x|)^2\,dx)$. In the paper of Fujishima and Ishige (2011) [8] the authors of this paper studied the behavior of the blow-up time and the blow-up set of the solution of (P) as $D \to \infty$ for the case $\int_{\mathbf{R}^N} \varphi(x)\,dx > 0$. In this paper, as a continuation of Fujishima and Ishige (2011) [8], we consider the case

$$\int_{\mathbf{R}^N} \varphi(x)\,dx \leqslant 0,$$

and study the behavior of the blow-up time and the blow-up set of the solution of (P) as $D \to \infty$. The behavior in the case $\int_{\mathbf{R}^N} \varphi(x)\,dx \leqslant 0$ is completely different from the one in the case $\int_{\mathbf{R}^N} \varphi(x)\,dx > 0$.

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1. Introduction

In this paper we are concerned with the Cauchy problem for a semilinear heat equation,

$$\partial_t u = D\Delta u + |u|^{p-1}u, \quad x \in \mathbf{R}^N,\ t > 0, \tag{1.1}$$

$$u(x,0) = \lambda + \varphi(x), \quad x \in \mathbf{R}^N, \tag{1.2}$$

where $\partial_t = \partial/\partial t$, $D > 0$, $p > 1$, $N \geqslant 3$, $\lambda > 0$, and

$$\varphi \in L^\infty\big(\mathbf{R}^N\big) \cap L^1\big(\mathbf{R}^N, \big(1+|x|\big)^2\,dx\big). \tag{1.3}$$

Let T_D be the maximal existence time of the unique classical solution u of (1.1) and (1.2). If $T_D < \infty$, then

$$\limsup_{t\to T_D} \sup_{x\in\mathbf{R}^N} \big|u(x,t)\big| = \infty,$$

and we call T_D the blow-up time of the solution u. Furthermore we denote by B_D the blow-up set of the solution u, that is,

$$B_D = \Big\{x \in \mathbf{R}^N\colon \text{there exists a sequence } \big\{(x_n,t_n)\big\} \subset \mathbf{R}^N \times (0,T_D)$$
$$\text{such that } \lim_{n\to\infty}(x_n,t_n) = (x,T_D),\ \lim_{n\to\infty}\big|u(x_n,t_n)\big| = +\infty\Big\}.$$

The blow-up set is an interesting subject for the study of the blow-up problem for the semilinear heat equation (1.1), and has been studied intensively by many mathematicians (see for example [2–12,14–16,18–20,22–27], and a survey [21], which includes a considerable list of references for this topic). Generally speaking, the location of the blow-up set is decided by given data such as the initial function and the boundary conditions and by the balance between the diffusion and the nonlinear term. In particular, if D is sufficiently large, then the behavior of the solution heavily depends on the diffusion driven from Laplacian Δ, and we can expect that the location of the blow-up set is decided by the diffusion term Δ. Indeed, as pointed out in [12,15], and [16], for the Cauchy–Neumann problem for the semilinear heat equation (1.1) in a bounded domain, the limit of the blow-up set B_D as $D \to \infty$ coincides with the limit of the hot spots of the solution of the heat equation as $t \to \infty$ and it is characterized as the set of the maximum points of the projection of the initial function onto the second Neumann eigenspace. In this paper we consider problem (1.1) and (1.2), and study the relationship between the behavior of blow-up time and the blow-up set as $D \to \infty$ and the large time behavior of the solution of the heat equation.

Let φ be a function satisfying (1.3). Then the function

$$\big(e^{t\Delta}\varphi\big)(x) := (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{-\frac{|x-y|^2}{4t}} \varphi(y)\,dy \tag{1.4}$$

is a unique bounded classical solution of the heat equation with the initial function φ. We denote by $M(\varphi)$ the mass of the function φ, that is,

$$M(\varphi) := \int_{\mathbf{R}^N} \varphi(x)\,dx,$$