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Global well-posedness for a modified critical dissipative quasi-geostrophic equation

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A B S T R A C T

In this paper we consider the following modified quasi-geostrophic equation

$$\partial_t\theta + u\cdot\nabla\theta + \nu|D|^\alpha\theta = 0, \qquad u = |D|^{\alpha-1}\mathcal{R}^\perp\theta, \quad x\in\mathbb{R}^2$$

with $\nu > 0$ and $\alpha \in]0,1[\,\cup\,]1,2[$. When $\alpha \in]0,1[$, the equation was firstly introduced by Constantin, Iyer and Wu (2008) in [11]. Here, by using the modulus of continuity method, we prove the global well-posedness of the system. As a byproduct, we also show that for every $\alpha \in]0,2[$, the Lipschitz norm of the solution has a uniform exponential upper bound.

1. Introduction

In this paper we focus on the following modified 2D dissipative quasi-geostrophic equation

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \nu|D|^\alpha\theta = 0, \\ u = |D|^{\alpha-1}\mathcal{R}^\perp\theta, \qquad \theta|_{t=0} = \theta_0(x) \end{cases} \tag{1.1}$$

with $\nu > 0$, $\alpha \in]0,1[\,\cup\,]1,2[$, $|D|^\beta = (-\Delta)^{\frac{\beta}{2}}$ is defined via the Fourier transform

$$\widehat{|D|^\beta f}(\zeta) = |\zeta|^\beta \hat{f}(\zeta)$$

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and

$$\mathcal{R}^{\perp}\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta) := |D|^{-1}(\partial_2\theta, -\partial_1\theta)$$

where $\mathcal{R}_i$ $(i = 1, 2)$ are the usual Riesz transforms (cf. [16]).

When $\alpha = 0$, this model describes the evolution of the vorticity of a two-dimensional damped inviscid incompressible fluid. The case of $\alpha = 1$ just is the critical dissipative quasi-geostrophic equation which arises in the geostrophic study of rotating fluids (cf. [8]). Although when $\alpha = 2$ the flow term in (1.1) vanishes, we can still view the model introduced in [17] as a meaningful generalization of this endpoint case, where the model is derived from the study of the full magnetohydrodynamic equations and the divergence-free three-dimensional velocity u satisfies $u = M[\theta]$ with M a nonlocal differential operator of order 1. We also refer to [3–5] for other related generalized quasi-geostrophic models.

For convenience, we here recall the well-known 2D quasi-geostrophic equation

$$(QG)_\alpha \quad \begin{cases} \partial_t\theta + u\cdot\nabla\theta + \nu|D|^\alpha\theta = 0, \\ u = \mathcal{R}^{\perp}\theta, \qquad \theta(0,x) = \theta_0(x), \end{cases}$$

where $\nu \geqslant 0$ and $0 \leqslant \alpha \leqslant 2$. When $\nu > 0$, $\alpha \in]0,1[\cup]1,2[$, we observe that the system (1.1) is almost the same with the quasi-geostrophic equation, and its only difference lies on introducing an extra $|D|^{\alpha-1}$ in the definition of u. When $\alpha \in]0,1[$, $|D|^{\alpha-1}$ is a negative derivative operator and always plays a good role; while when $\alpha \in]1,2[$, $|D|^{\alpha-1}$ is a positive derivative operator and always takes a bad part. Moreover, corresponding to the dissipation operator $|D|^\alpha$ in the equation $(QG)_\alpha$, this additional operator makes it be a new balanced state: the flow term $u\cdot\nabla\theta$ scales the same way as the dissipative term $|D|^\alpha\theta$, i.e., Eq. (1.1) is scaling invariant under the transformation

$$\theta(t,x) \mapsto \theta_\lambda(t,x) := \theta\big(\lambda^\alpha t, \lambda x\big), \quad \text{with } \lambda > 0.$$

We note that in the sense of scaling invariance, the system (1.1) is similar to the critical quasi-geostrophic equation $(QG)_1$.

Recently, when $\alpha \in]0,1[$, Constantin, Iyer and Wu in [11] introduced this modified quasi-geostrophic equation and proved the global regularity of Leray–Hopf weak solutions to the system with L^2 initial data. Basically, they use the method from Caffarelli–Vasseur [2] which deals with the same issue of 2D critical quasi-geostrophic equation $(QG)_1$. We also remark that partially because of its simple form and its internal analogy with the 3D Euler/Navier–Stokes equations, the quasi-geostrophic equation $(QG)_\alpha$, especially the critical one $(QG)_1$, has been extensively considered (see e.g. [1–3,7–10,12,14,18,23] and references therein). While global regularity of Navier–Stokes equations remains an outstanding challenge in mathematical physics, the global issue of the 2D critical dissipative quasi-geostrophic equation has been in a satisfactory state. In [10], Constantin, Cordoba and Wu showed the global well-posedness of the classical solution under the condition that the zero-dimensional L^∞ norm of the data is small. This smallness assumption was firstly removed by Kiselev, Nazarov and Volberg in [18], where they obtained the global well-posedness for the arbitrary periodic smooth initial data by using a modulus of continuity method. Almost at the same time, Caffarelli and Vasseur in [2] resolved the problem to establish the global regularity of weak solutions associated with L^2 initial data by exploiting the De Giorgi method. We also cite the work of Abidi and Hmidi [1] and Dong and Du [14], as extended work of [18], in which the authors proved the global well-posedness with the initial data belonging to the (critical) space $\dot{B}^0_{\infty,1}$ and H^1 respectively without the additional periodic assumption.

The main goal in this paper is to prove the global well-posedness of the system (1.1) with $\alpha \in]0,1[\cup]1,2[$. In contrast with the work of [11], we here basically follow the pathway of [18] to obtain the global results by constructing suitable moduli of continuity. Precisely, we have

Theorem 1.1. *Let $\nu > 0$, $\alpha \in]0,2[$ and $\theta_0 \in H^m(\mathbb{R}^2)$, $m > 1$, then there exists a unique global solution*

$$\theta \in \mathcal{C}\big([0,\infty[; H^m\big) \cap L^2_{\mathrm{loc}}\big([0,\infty[; H^{m+\frac{\alpha}{2}}\big) \cap \mathcal{C}^\infty\big(]0,\infty[\times\mathbb{R}^2\big)$$