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Multiple solutions for a Hénon-like equation on the annulus

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Abstract

For the equation $-\Delta u = ||x| - 2|^{\alpha} u^{p-1}$, $1 < |x| < 3$, we prove the existence of two solutions for α large, and of two additional solutions when p is close to the critical Sobolev exponent $2^* = 2N/(N-2)$. A symmetry-breaking phenomenon appears, showing that the least-energy solutions cannot be radial functions.

MSC: 35J40

Keywords: Symmetry breaking; Hénon-like equation

1. Introduction

In this paper we will consider the following problem:

$$\begin{cases} -\Delta u = \Psi_{\alpha} u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Omega = \{x \in \mathbb{R}^N \mid 1 < |x| < 3\}$ is an annulus in $\mathbb{R}^N$, $N \geqslant 3$, $\alpha > 0$, $p > 2$ and Ψ_α is the radial function

$$\Psi_\alpha(x) = \big||x| - 2\big|^\alpha.$$

This equation can be seen as a natural extension to the annular domain Ω of the celebrated Hénon equation with Dirichlet boundary conditions (see [9,11])

$$\begin{cases} -\Delta u = |x|^\alpha |u|^{p-1} & \text{for } |x| < 1, \\ u = 0 & \text{if } |x| = 1. \end{cases} \tag{2}$$

Actually, the weight function Ψ_α reproduces on Ω a similar qualitative behavior of $|\cdot|^\alpha$ on the unit ball B of $\mathbb{R}^N$.

A standard compactness argument shows that the infimum

$$\inf_{\substack{u \in H_0^1(B) \\ u \neq 0}} \frac{\int_B |\nabla u|^2\,dx}{(\int_B |x|^\alpha |u|^p\,dx)^{2/p}} \tag{3}$$

is achieved for any $2 < p < 2^*$ and any $\alpha > 0$. In 1982, Ni proved in [11] that the infimum

$$\inf_{\substack{u \in H_{0,\mathrm{rad}}^1(B) \\ u \neq 0}} \frac{\int_B |\nabla u|^2\,dx}{(\int_B |x|^\alpha |u|^p\,dx)^{2/p}} \tag{4}$$

is achieved for any $p \in (2, 2^* + \frac{2\alpha}{N-2})$ by a function in $H_{0,\mathrm{rad}}^1(B)$, the space of radial $H_0^1(B)$ functions. Thus, radial solutions of (2) exist also for (Sobolev) supercritical exponents p. Actually, radial H_0^1 elements show a power-like decay away from the origin (as a consequence of the Strauss Lemma, see [1,18]) that combines with the weight $|x|^\alpha$ and provides the compactness of the embedding $H_{0,\mathrm{rad}}^1(B) \subset L^p(B)$ for any $2 < p < 2^* + \frac{2\alpha}{N-2}$.

A natural question is whether any minimizer of (3) must be radially symmetric in the range $2 < p < 2^*$ and $\alpha > 0$. Since the weight $|\cdot|^\alpha$ is an increasing function, neither rearrangement arguments nor the moving plane techniques of [8] can be applied.

Reverting to the case $\alpha > 0$, Smets et al. proved in [16] some symmetry-breaking results for (2). They proved that minimizers of (3) (the so-called *ground-state solutions*, or least energy solutions) cannot be radial, at least for α large enough. As a consequence, (2) has at least two solutions when α is large (see also [17]).

Later on, Serra proved in [15] the existence of at least one nonradial solution to (2) in the critical case $p = 2^*$, and in [2] the authors proved the existence of more than one solution to the same equation also for some supercritical values of p. These solutions are nonradial and they are obtained by minimization under suitable symmetry constraints.

Quite recently, Cao and Peng proved in [7] that, for p sufficiently close to 2^*, the ground-state solutions of (2) possess a unique maximum point whose distance from ∂B tends to zero as $p \to 2^*$.

This kind of result was improved in [13], where multibump solutions for the Hénon equation with almost critical Sobolev exponent p are found, by means of a finite-dimensional reduction. These solution are not radial, though they are invariant under the action of suitable subgroups of