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ScienceDirect
J. Differential Equations 245 (2008) 1526–1565
Journal of Differential Equations
www.elsevier.com/locate/jde

Center manifolds for periodic functional differential equations of mixed type

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Received 23 July 2007; revised 20 February 2008
Available online 10 June 2008

Abstract

We study the behaviour of solutions to nonlinear functional differential equations of mixed type (MFDEs), that remain sufficiently close to a prescribed periodic solution. Under a discreteness condition on the Floquet spectrum, we show that all such solutions can be captured on a finite dimensional invariant center manifold, that inherits the smoothness of the nonlinearity. This generalizes the results that were obtained previously in [H.J. Hupkes, S.M. Verduyn Lunel, Center manifold theory for functional differential equations of mixed type, J. Dynam. Differential Equations 19 (2007) 497–560] for bifurcations around equilibrium solutions to MFDEs.

Keywords: Mixed type functional differential equation; Floquet theory; Center manifold; Hopf bifurcation; Finite dimensional reduction; Advanced and retarded arguments

1. Introduction

In this paper we provide a tool to analyze the behaviour of solutions to a functional differential equation of mixed type (MFDE),

$$\dot{x}(\xi) = G(x_\xi), \tag{1.1}$$

that lie in the vicinity of a prescribed periodic solution. Here x is a continuous $\mathbb{C}^n$-valued function and for any $\xi \in \mathbb{R}$, the state $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$. We allow

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.02.026

$r_{\min} \leqslant 0$ and $r_{\max} \geqslant 0$, hence the operator G may depend on advanced and retarded arguments simultaneously.

Historically, the primary motivation for the study of MFDEs comes from the study of lattice differential equations (LDEs), which are systems of differential equations indexed by points on an infinite spatial lattice, e.g. $h\mathbb{Z}^n$ for some integer n and grid size $h > 0$. Such equations allow the incorporation of nonlocal interactions into otherwise local models and exhibit richer behaviour than the limiting PDEs that arise when taking $h \to 0$ [6,11,17,24,25]. For these reasons, models involving LDEs have been developed in many scientific disciplines, including chemical reaction theory [12,21], image processing and pattern recognition [7], material science [3,5] and biology [2,4,20]. As a specific example which is interesting in view of our main equation (1.1), we recall a Frenkel–Kontorova type model that was analyzed numerically in [1]. This model was originally developed to describe the motion of dislocations in a crystal [29,30], but now has numerous other applications in the literature. In particular, consider a chain of particles that have positions x_k, with $k \in \mathbb{Z}$. The dynamics are given by the LDE

$$\ddot{x}_k(t) + \gamma \dot{x}_k(t) = x_{k-1}(t) - 2x_k(t) + x_{k+1}(t) - d \sin x_k(t) + F, \tag{1.2}$$

in which γ and d are parameters and F is an external applied force. In the literature a special class of travelling wave solutions, which have been named uniform sliding states, has been constructed for (1.2). Such solutions can be written in the form $x_k(t) = \phi(k - ct)$ for some waveprofile ϕ and wavespeed c and in addition satisfy the special condition $x_{k+N} = x_k + 2\pi M$, in which N and M are fixed integers. It is not hard to see that (1.2) can be restated in such a way that these states become periodic and hence the study of bifurcations from these solutions can be fitted into the framework developed here.

Recently [18], based upon earlier work by several authors [9,23,32], a center manifold approach was developed to capture all solutions of (1.1) that remain sufficiently close to a given equilibrium $\bar{x}$. It was shown that the dimension and linear structure on the center manifold are entirely determined by the holomorphic characteristic matrix $\Delta : \mathbb{C} \to \mathbb{C}^{n \times n}$ associated to the linearized system $\dot{v}(\xi) = DG(\bar{x})v_\xi$. This matrix is explicitly given by $\Delta(z) = zI - DG(\bar{x})\exp(z\cdot)$ and is thus relatively straightforward to construct and analyze in many practical applications, see e.g. [8,15]. As an illustration of the strength of this reduction, consider a parameter dependent family of MFDEs,

$$\dot{x}(\xi) = G(x_\xi, \mu), \tag{1.3}$$

that admit a common equilibrium $\bar{x}$. In addition, suppose that a pair of roots of the characteristic equation $\det \Delta(z, \mu) = 0$ crosses the imaginary axis at a certain parameter value μ_0. Under suitable conditions the Hopf bifurcation theorem can be lifted to the infinite dimensional setting of (1.3) and hence one may conclude the existence of a branch of periodic solutions to (1.3) bifurcating from the equilibrium $\bar{x}$ for $\mu \sim \mu_0$. In [16] this approach was used to analyze an economic optimal control problem involving delays. This problem was proposed by Rustichini in order to simplify a model describing the dynamics of a capital market [27], whilst still retaining the periodic orbits that are compulsory for any such model. The existence of these periodic orbits was established by numerically analyzing the resulting characteristic equation and looking for root-crossings through the imaginary axis.