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Asymptotically linear Schrödinger equation with potential vanishing at infinity

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Abstract

We are concerned with the existence of bound states and ground states of the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u(x) + V(x)u(x) = K(x)f(u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u(x) > 0, & N \geqslant 3, \end{cases} \tag{0.1}$$

where the potential $V(x)$ may vanish at infinity, $f(s)$ is asymptotically linear at infinity, that is, $f(s) \sim O(s)$ as $s \to +\infty$. For this kind of potential, it seems difficult to find solutions in $H^1(\mathbb{R}^N)$, i.e. bound states of (0.1). If $f(s) = s^p$ and $p \in (\sigma, (N+2)/(N-2))$ with $\sigma \geqslant 1$, Ambrosetti, Felli and Malchiodi [A. Ambrosetti, V. Felli, A. Malchiodi, Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. Eur. Math. Soc. 7 (2005) 117–144] showed that (0.1) has a solution in $H^1(\mathbb{R}^N)$ and (0.1) has no ground states if p is out of the above range. In this paper, we are interested in what happens if $f(s)$ is asymptotically linear. Under appropriate assumptions on K, we prove that (0.1) has a bound state and a ground state.

MSC: 35J20; 35J60

Keywords: Nonlinear Schrödinger equation; Vanishing potential; Bound state

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1. Introduction

In this paper, we consider the following stationary nonlinear Schrödinger equation

$$-\Delta u(x) + V(x)u(x) = K(x)f(u), \quad x \in \mathbb{R}^N,\ N \geqslant 3, \tag{1.1}$$

where the functions V, f and K satisfy the following conditions:

(V_1) $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and there exist $a, A, \alpha > 0$ such that

$$\frac{a}{1+|x|^\alpha} \leqslant V(x) \leqslant A.$$

(F_1) $f \in C(\mathbb{R}, \mathbb{R}^+)$, $f(s) \equiv 0$ for all $s \leqslant 0$ and $f(s)s^{-1} \to 0$ as $s \to 0^+$.
(F_2) There exists $l \in (0, +\infty)$ such that $f(s)s^{-1} \to l$ as $s \to \infty$.
(K_1) K is a positive continuous function and there exists $R_0 > 0$ such that

$$\sup\left\{\frac{f(s)}{s}\colon s > 0\right\} < \inf\left\{\frac{V(x)}{K(x)}\colon |x| \geqslant R_0\right\}.$$

It is easy to see that the condition (K_1) can be obtained by assuming that

$$\sup\left\{\frac{f(s)}{s}\colon s > 0\right\} < \liminf_{|x|\to+\infty}\left\{V(x)/K(x)\right\}. \tag{1.2}$$

Remark 1.1. In paper [1], the condition (V_1) is also assumed, but the condition on K is as follows

(K_2) $K : \mathbb{R}^N \to \mathbb{R}$ is smooth and there exist $k, \beta > 0$ such that

$$0 < K(x) \leqslant \frac{k}{1+|x|^\beta}. \tag{1.3}$$

Clearly, if (K_2) holds with $0 < \alpha < \beta$, then $\lim_{|x|\to+\infty} V(x)/K(x) = +\infty$. By (1.2), we see that the condition (K_2) with $\alpha \in (0, \beta)$ leads to the condition (K_1).

Here are two examples, in which our conditions (V_1), (F_1), (F_2) and (K_1) are satisfied, but (K_2) fails.

Example 1.1. Let $V(x) = \frac{1}{\ln\ln(3+|x|)}$ and $K(x) = \frac{1}{\ln(3+|x|^2)}$, we see that ($V_1$) holds and $\lim_{|x|\to+\infty} \frac{V(x)}{K(x)} = +\infty$, then it is easy to see there is f such that (F_1), (F_2) and (K_1) hold. But in this case, K does not satisfy (K_2).

Example 1.2. For any fixed $R_0 > 0$, let $V(x) = 1/\ln(3 + |x|)$ and

$$K(x) = \begin{cases} V(x)/(1+|x|), & \text{if } |x| < R_0, \\ V(x)/(1+R_0), & \text{if } |x| \geqslant R_0, \end{cases} \qquad f(s) = \begin{cases} R_0 s^2/(1+s), & \text{if } s > 0, \\ 0, & \text{if } s \leqslant 0. \end{cases}$$