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Persistence and imperfection of nonautonomous bifurcation patterns

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A B S T R A C T

For nonautonomous dynamical systems a bifurcation can be understood as topological change in the set of bounded entire solutions to a given time-dependent evolutionary equation. Following this idea, a Fredholm theory via exponential dichotomies on semi-axes enables us to employ tools from analytical branching theory yielding nonautonomous versions of fold, transcritical and pitchfork patterns. This approach imposes the serious hypothesis that precise quantitative information on the dichotomies is required — an assumption hard to satisfy in applications. Thus, imperfect bifurcations become important.

In this paper, we discuss persistence and changes in the previously mentioned bifurcation scenarios by including an additional perturbation parameter. While the unperturbed case captures the above bifurcation patterns, we obtain their unfolding and therefore the local branching picture in a whole neighborhood of the system. Using an operator formulation of parabolic differential, Carathéodory differential and difference equations, this will be achieved on the basis of recent abstract analytical techniques due to Shi (1999) and Liu, Shi and Wang (2007).

1. Introduction

1.1. Motivation

By definition, a local theory of dynamical systems deals with the behavior of difference or differential equations in the vicinity of invariant sets like equilibria or periodic solutions. As soon as the equations of interest become aperiodically time-dependent, however, usually neither equilibria nor

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periodic solutions exist. This turns out to be problematic and challenging, since such nonautonomous problems are omnipresent in a multitude of applications where modulation, control or even random effects cannot be neglected. Thus, the question arises which invariant objects are appropriate to establish a suitable nonautonomous bifurcation theory?

In this regard, it was observed that equilibria generically persist as bounded entire solutions under small temporally fluctuating perturbations (see, for instance, [6,36]). More general this behavior holds for so-called *hyperbolic* entire solutions, whose corresponding variational equation admits an exponential dichotomy on the whole time axis. On this basis, it is reasonable to replace equilibria by bounded entire solutions as natural bifurcating objects in a time-varying framework. Furthermore, related nonautonomous problems also occur in a purely autonomous setting when one is interested in the behavior near aperiodic reference solutions and their behavior under varying parameters (or even equations).

We considered this as motivation and starting point to investigate the bifurcation behavior of bounded entire solutions in [33,35] using tools from analytical branching theory (cf. [17] or [43, Chapter 8]), like Lyapunov–Schmidt reduction. The required Fredholm theory is provided by means of dynamical properties for the variational equation along a *nonhyperbolic* reference solution. This enabled us to derive nonautonomous versions of the classical fold, transcritical and pitchfork bifurcation patterns in [33]. Furthermore, a crossing curve bifurcation (generalizing transcritical and pitchfork patterns) and a degenerate fold bifurcation have been obtained in [35] on the basis of abstract analytical results due to [27]. Keeping in mind that a (global) pullback attractor $\mathcal{A}$ (cf. e.g. [18]) of a nonautonomous dynamical system consists of bounded entire solutions, the foregoing bifurcation concept has also stringent consequences on the structure of $\mathcal{A}$ and the resulting notion of attractor bifurcation as investigated in [37, pp. 42ff, Section 2.5] or [19,24,38]. Yet, extending these preparations and earlier approaches, the motivation for our present work is two-fold:

First, one problem in our previous approach is that it essentially requires very detailed information on the exponential dichotomy data (the invariant projectors), as well as hypotheses on the whole time axis. In practice, such conditions can be verified only numerically or approximately. Despite yielding very precise information on the local structure of the set of bounded solutions, the results of [33, 35] are therefore somewhat academic. This gives rise to the natural question for the behavior of bifurcation scenarios under perturbation yielding so-called imperfect bifurcations: What is the actual bifurcation diagram for systems in a neighborhood of the bifurcating one. Second, we like to investigate nonautonomous bifurcations under external perturbations, which can be small, but otherwise arbitrary bounded fluctuations. In doing so, we give an accurate description on how the structurally unstable scenario of a nonautonomous bifurcation gets destroyed under perturbation.

Throughout we are interested in the behavior of evolutionary equations depending on a real bifurcation parameter λ, which for given bifurcation value $\lambda = \lambda^*$ possesses a bounded entire reference solution ϕ^*. This solution is supposed to be nonhyperbolic in the sense that the corresponding variational equation has 0 in its dichotomy spectrum (cf. [39,42]), respectively 1 in the discrete case (see [2]). More precisely, it admits exponential dichotomies on both the positive and the negative semiaxes, whose projectors do not span the whole state space (cf. (2.7)). This is an intrinsically nonautonomous form of nonhyperbolicity and cannot occur for almost-periodic, periodic or autonomous equations. Thus, certain natural nonautonomous bifurcation scenarios are not covered by our abstract approach. Yet we believe to make a valid contribution being complementary to prior nonautonomous bifurcation scenarios of e.g. [19,24,38].

Our semiaxes dichotomy assumption requires the evolutionary equations to be at least two-dimensional and the bifurcating solution ϕ^* to be unstable. Such one-sided dichotomies guarantee the existence of a stable integral manifold $\mathcal{W}_\lambda^+$ (consisting of forward bounded solutions), and of an unstable integral manifold $\mathcal{W}_\lambda^-$ (which contains the backward bounded solutions). Our bifurcation notion is based on the structure of all entire bounded solutions near ϕ^* and therefore, a bifurcation is a topological change in the intersection $\mathcal{W}_\lambda^+ \cap \mathcal{W}_\lambda^-$ for varying parameters λ. In fact, this intersection yields initial values for bounded entire solutions (cf. Fig. 1) and allows a vivid illustration of our bifurcation scenarios. Nonetheless, since $\mathcal{W}_\lambda^+$ and $\mathcal{W}_\lambda^-$ are not explicitly known, our approach is purely analytical and the fundamental results of [41,27] show that